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Martingales, nonstationary increments, and the efficient market hypothesis

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Abstract

We discuss the deep connection between nonstationary increments, martingales, and the efficient market hypothesis for stochastic processes $x(t)$ with arbitrary diffusion coefficients $D(x, t)$. We explain why a test for a martingale is generally a test for uncorrelated increments. We explain why martingales look Markovian at the level of both simple averages and 2-point correlations. But while a Markovian market has no memory to exploit and cannot be beaten systematically, a martingale admits memory that might be exploitable in higher order correlations. We also use the analysis of this paper to correct a misstatement of the 'fair game' condition in terms of serial correlations in Fama's paper on the EMH. We emphasize that the use of the log *increment* as a variable in data analysis generates spurious fat tails and spurious Hurst exponents.

Keywords: Martingales; Markov processes; Memory; Stationary and nonstationary increments; Autocorrelations; Efficient market hypothesis; Fractional Brownian motion

1. Introduction

The main point of this paper is to explain why in foreign exchange (FX) and, in many other data analyses as well, neither the log *increment* $x(t; T) = \ln p(t + T)/p(t)$ nor the price difference $\Delta p = p(t + T) - p(t)$ can be used to deduce the correct 1-point *log returns* density. By the log returns density $f_1(x, t)$ we mean the histograms obtained from FX time series for the log return $x(t) = \ln(p(t)/p_c)$ where $p(t)$ is a price at time t and $p_c(t)$ is a reference price that can be understood as 'value' [1]. The 'consensus price' p_c is simply the price that locates the peak of the 1-point returns density $f_1(x, t)$ at time t.

In a process with stationary increments [2] $x(t, T)$, meaning that the density of increments $f(x, t, t + T)$ is independent of the starting time t, the increment $x(t, T) - x(t + T) - x(t) = x(T)$ is a 'good' variable with 1-point density $f_1(x, T)$. If the variance is nonlinear in the time t, then such processes *necessarily* have the long time increment autocorrelations exemplified by fractional Brownian motion (fBm) [3] and so violate the efficient market

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hypothesis (EMH) at the level of pair correlations. We know that the EMH is a good zeroth order approximation to real FX markets, because FX markets are hard to beat, and the observed FX market variance is strongly nonlinear [4]. Detrended, such markets are described by martingale stochastic processes [5], and martingales generally generate nonstationary increments. Here is one main point: *If one applies the assumption of stationary increments to a time series with nonstationary ones and nonlinear variance, then one generates as artifacts both spurious fat tails and a spurious Hurst exponent* $H_S = 1/2$ [4]. Because of this, most existing FX data analyses in the literature are wrong. That mistake is equivalent to assuming that the 'log increment' is a good variable in data analysis.

So, by 'log return' we generally understand that the variable is $x(t) = \ln(p(t)/p_c)$ and not the log increment $x(t, T) = \ln p(t+T)/p(t)$. This fact is connected intimately with the notion of a martingale: martingales generally require processes with nonstationary increments [5]. One may take the martingale condition as the basic definition of the EMH, because in that case memory at easy levels of detection is ruled out (a martingale behaves Markovian at the level of averages and 2-point correlations), but memory at harder to detect levels of correlation is permitted [5]. Contrary to nearly all assertions in the existing literature, *scaling does not matter*, it is only the question of stationary vs. nonstationary increments that determines the presence or lack of long time increment autocorrelations [1,5]. *For processes with nonlinear variance, the use of the increment* $x(t, T) = \ln p(t+T)/p(t)$ *to build histograms would be correct iff. the EMH would be systematically violated in a very specific way. The typical data analysis uses a technique called 'sliding windows' that implicitly assumes that the distribution of* $x(t, T)$ *is independent of* t [4,5].

2. Stationary vs. nonstationary increments

We assume that $[-\infty < x < \infty\}$ so that with only trivial drift terms stochastic processes cannot approach statistical equilibrium [6]. All processes considered are therefore nonstationary ones. Stationary increments are defined by

$$x(t+T) - x(t) = x(T), \tag{1}$$

'in distribution', and by nonstationary increments [2–5] we mean that

$$x(t+T) - x(t) \neq x(T) \tag{2}$$

in distribution. When (1) holds, then given the density of 'positions' $f_1(x, t)$, we also know the density $f_1(x(T), T) = f_1(x(t+T) - x(t), T)$ of increments independently of the the starting time t. Whenever the increments are nonstationary then any analysis of the increments requires the 2-point density, $f_2(x(t+T), t+T; x(t), t)$. Here, the 1-point increments density depends on the staring time t: let $z = x(t; T)$. Then

$$f(z, t, t+T) = \int f_2(y, t+T; x, t)\delta(z - y + x)\mathrm{d}x\mathrm{d}y \tag{3}$$

will not be not independent of t, although attempts to construct this quantity as histograms in data analysis via 'sliding windows' implicitly presume t-independence [4,7].

According to Mandelbrot [8], a so-called 'efficient market' has no memory that can be *easily* exploited in trading. We assume that the market is hard but not necessarily impossible to beat and we restrict ourselves to normal liquid markets, ruling out both crashes and big trades that cause liquidity to dry up [6,9]. We then take as the necessary but not sufficient condition for an impossible to beat market the absence of increment autocorrelations,

$$\langle (x(t_1) - x(t_1 - T_1))(x(t_2 + T_2) - x(t_2)) \rangle = 0, \tag{4}$$

when there is no time interval overlap, $t_1 < t_2$ and T_1, $T_2 > 0$. This is a much weaker condition and leads to far more interesting market dynamics than would occur were the increments merely statistically independent. We will see that this condition largely determines the class of dynamics (martingales), and rules out processes with increment autocorrelations due to stationary increments like fBm [3,5]. This also eliminates processes with correlated nonstationary increments like the time translationally invariant Gaussian transition densities used in statistical physics [10].

Consider a drift-free stochastic process $x(t)$ where the increments are uncorrelated. From this condition we easily obtain the autocorrelation function for positions (returns), sometimes called 'serial autocorrelations': If $t > s$ then (4) yields

$$\langle x(t)x(s) \rangle = \langle (x(t) - x(s))x(s) \rangle + \langle x^2(s) \rangle = \langle x^2(s) \rangle > 0, \tag{5}$$

since with $x(t_o) = 0$, $x(s) - x(t_o) = x(s)$, so that $\langle x(s)x(t)\rangle = \langle x^2(s)\rangle$ is simply the variance in x. Given a history $(x(t), \ldots, x(s), \ldots, x(0))$, or $(x(t_n), \ldots x(t_k), \ldots, x(t_1))$, (4) reflects the martingale property. With $p_n(x_n, t_n|x_{n-1}, t_{n-1}; \ldots; x_1, t_1)$ denoting the 2-point conditional probability density depending on $n-1$ points $(x_{n-1}, t_{n-1}; \ldots; x_1, t_1)$ in the past [5], then

$$\begin{aligned}\langle x(t_n)x(t_k)\rangle &= \int \mathrm{d}x_n \ldots \mathrm{d}x_1 x_n x_k p_n(x_n, t_n|x_n, t_n, \ldots, x_n, t_n, \ldots) p_{n-1}(\ldots) \ldots p_{k+1}(\ldots) f_k(\ldots) \\ &= \int x_k^2 f_k(x_k, t_k; \ldots; x_1, t_1)\mathrm{d}x_k \ldots \mathrm{d}x_1 = \int x^2 f_1(x, t)\mathrm{d}x = \langle x_k^2(t_k)\rangle \end{aligned} \tag{6}$$

where

$$\int x_m \mathrm{d}x_m p_m(x_m, t_m|x_{m-1}, t_{m-1}; \ldots; x_1, t_1) = x_{m-1} \tag{7}$$

because, starting with Eq. (7), and taking the absolute average to form the autocorrelation function, we obtain Eq. (6). Every martingale generates uncorrelated increments and conversely, and so for a Martingale $\langle x(t)x(s)\rangle = \langle x^2(s)\rangle$ if $s < t$.

In a martingale process, the history dependence cannot be detected at the level of 2-point correlations, memory effects can at best first appear at the level 3-point correlations, requiring the study of higher order transition densities. Here, we have not postulated a martingale, instead we have deduced that property from the lack of pair wise increment correlations. But this is only part of the story. What follows next is crucial for avoiding mistakes in data analysis.

Combining

$$\langle (x(t+T) - x(t))^2\rangle = +\langle (x^2(t+T))\rangle + \langle x^2(t)\rangle - 2\langle x(t+T)x(t)\rangle \tag{8}$$

with (29), we get

$$\langle (x(t+T) - x(t))^2\rangle = \langle x^2(t+T)\rangle - \langle x^2(t)\rangle \tag{9}$$

which depends on *both* t and T, excepting the case where $\langle x^2(t)\rangle$ is linear in t. Uncorrelated increments are generally nonstationary [5,11]. ***Therefore, martingales generate uncorrelated, typically nonstationary increments.*** So, at the level of pair correlations a martingale with memory cannot be distinguished empirically from a drift-free Markov process. The increments of a martingale may be stationary iff. the variance is linear in t.

We have emphasized earlier [3] that stationary increments $x(t, T) = x(t+T) - x(t) = x(T)$ with finite variance $\langle x^2(t)\rangle < \infty$ generate the long time increment autocorrelations characteristic of fBm [3,5], whereas stationary uncorrelated increments with infinite variance occur in Levy processes [12,13]. Stationary Gaussian processes with correlated nonstationary increments are implicit in the models of Ref. [10]. Many interesting properties of martingales are derived in Ref. [14].

3. The efficient market hypothesis

In our opinion the EMH is simply the attempt to mathematize the idea that normal, liquid, finance markets are very hard to beat. If there is no useful information in market prices, then those prices can be understood as noise, the product of 'noise trading' [6,15]. A martingale formulation of the EMH embodies the idea that the market is hard to beat, is overwhelmingly noise, but leaves open the question of hard to find correlations that might be exploited for exceptional profit [5,8]. Our recent data analysis, using a 6 year string of Euro/Dollar data, establishes that detrended FX data over the past six years have uncorrelated nonstationary increments in log returns after 10 min. of trading, and can further be understood a martingale with complicated nonlinear variance in the log return variable x over the time interval of a day or a week.

A strict interpretation of the EMH is that there are no correlations, no patterns of any kind, that can be employed *systematically* to beat the average return $\langle R\rangle$ reflecting the market itself. A Markov market is in principle unbeatable, it has no systematically repeated patterns, no memory to exploit. We will argue below that the stipulation should be added that in discussing the EMH we should consider only normal, liquid markets, meaning very liquid markets with small enough transactions that approximately reversible trading is possible on a time scale of seconds [3]. Otherwise, 'Brownian-style' market models do not apply. Liquidity, 'the money bath' created by the noise traders whose behavior is reflected in the diffusion coefficient [9], is somewhat qualitatively analogous to the idea of the heat bath in thermodynamics, although the money bath is far from equilibrium and so cannot be described by thermodynamics [6]: the second by second fluctuations in $x(t)$ are created by the continual noise trading [9,15].

The martingale formulation of the EMH reflects the fact that financial markets are hard to beat, but leaves open the question whether the market might be beatable in principle at some higher level of correlation than pair correlations. Martingales admit finite memory [16,17] but that memory cannot be easily exploited to beat the market, precisely because the expectation of a martingale process $x(t)$ at any later time is simply the last observed return. The idea that memory may arise (in commodities, e,g.) from other unpredictable variables like the weather [8] or terrorism corresponds in statistical physics [18] to the appearance of memory as a consequence of averaging over other, more rapidly changing, variables in a larger dynamical system.

Understanding the EMH as a martingale condition on log returns is interesting because technical traders explicitly assume that certain price sequences give signals either to sell or buy. In principle, such memory is permitted in a martingale even if the market looks efficient. A particular price sequence $(p(t_n), \ldots, p(t_1))$, were it quasi-systematically to repeat, can be encoded as returns $(x_n, \ldots, x_1)$ so that a conditional probability density $p_n(x_n : x_{n-1}, \ldots, x_1)$ could be interpreted as a providing a risk measure to buy or sell. By 'quasi-repetition' of the sequence we mean that $p_n(x_n : x_{n-1}, \ldots, x_1)$ is significantly greater than a Markovian prediction. Typically, technical traders make the mistake of trying to interpret random price sequences quasi-deterministically, which differs from our interpretation of 'technical trading' based on conditional probabilities (see Lo et al. [19] for a discussion of technical trading claims, but based on a nonmartingale, nonempirically based model of prices). With only a conditional probability for 'signaling' a specific price sequence, an agent with a large debt to equity ratio can easily suffer the Gamblers' Ruin. In any case, we can offer no advice about technical trading, because the existence of market memory has not been established (the question is left open by the analysis of Ref. [19]), liquid finance markets look very Markovian so far as we have been able to understand the data [4], but one must go systematically beyond the level of pair correlations to try to find memory and there is no cookbook recipe to help us. Memory might, e.g., occur temporarily due to heavy trading around a particular price but could then be forgotten as other (even misleading) events occur.

Fama [20] took Mandelbrot's martingale as EMH proposal seriously and proposed to test finance data at the simplest level for a fair game condition. We now correct a mathematical mistake made by Fama (see the first two of three unnumbered equations at the bottom of pg. 391 in Ref. [20]), who wrongly concluded in his discussion of martingales as a fair game condition that $\langle x(t+T)x(t)\rangle = 0$. Here is his argument, rewritten partly in our notation. Let $x(t)$ denote a 'fair game'. With the initial condition chosen as $x(t_o) = 0$, then we have the unconditioned expectation $\langle x(t)\rangle = \int x \mathrm{d}x f_1(x,t) = 0$ (there is no drift). Then the so-called 'serial covariance' is given by

$$\langle x(t+T)x(t)\rangle = \int x \mathrm{d}x \langle x(t+T)\rangle_{\mathrm{cond}(x)} f_1(x,t). \tag{10}$$

Fama states that this autocorrelation vanishes because $\langle x(t+T)\rangle_{\mathrm{cond}} = 0$. This is impossible: by a fair game we mean a Martingale, the conditional expectation is $\langle x(t+T)\rangle_{\mathrm{cond}} = \int y \mathrm{d}y p_2(y, t+T; x, t) = x = x(t) \neq 0$, and so Fama should have concluded instead that $\langle x(t+T)x(t)\rangle = \langle x^2(t)\rangle$ as we showed in the last section. Vanishing of (10) would be true of statistically independent variables but is violated by a 'fair game'. Can Fama's argument be salvaged? Suppose that instead of $x(t)$ we would try to use the *increment* $x(t,T) = x(t+T) - x(t)$ as variable. Then $\langle x(t,T)x(t)\rangle = 0$ for a Martingale, as we showed in part 4. However, Fama's argument still would not be generally correct because $x(t,T)$ cannot be taken as a 'fair game' variable unless the variance is linear in t, and in financial markets the variance is not linear in t [4]. Fama's mislabeling of time dependent averages (typical in economics and finance literature) as 'market equilibrium' has been corrected elsewhere [6].

We do not follow the economists' tradition of trying to define three separate forms (weak, semi-strong, and strong of the EMH, where a nonfalsifiable distinction is made between three separate classes of traders. We specifically consider only normal liquid markets with trading times at multiples of 10 min. intervals so that a Martingale condition holds [4]. Normal market statistics overwhelmingly (with high probability, if not 'with measure one') reflect the noise traders [5], so we consider *only* normal liquid markets and ask whether noise traders produce signals that one might be able to trade on systematically. The question whether insiders or exceptional traders like Buffett and Soros can beat the market systematically probably cannot be tested scientifically: even if we had statistics on such exceptional traders, those statistics would likely be too sparse to draw a firm conclusion (see Ref. [4] for a discussion of the difficulty of getting good enough daily statistics on the noise traders, who dominate normal liquid markets). Furthermore, the exceptional traders apparently do not beat normal liquid markets, a high degree of illiquidity seems to play a significant role in their buy-low/sell-high successes. Effectively, or with high probability, there is only one type trader under consideration in Brownian market models (efficient market models), the noise trader. Noise traders provide the liquidity [6,15], their trading determines the form of the diffusion coefficient $D(x, t; \{x\})$, where $\{x\}$ reflects any memory present.

One can test for martingales and for violations of the EMH at increasing levels of correlation. At the level $n = 1$, the level of simple averages, the ability to detrend data implies a Martingale [4,5]. At the level $n = 2$, vanishing increment autocorrelations [5] implies a martingale. Both conditions are consistent with Markov processes and with the EMH. A positive test for a martingale *with memory* at the level $n \geqslant 3$ would eliminate Markov processes, and would violate the EMH as well. If such correlations exist and could be traded on then a typical finance theorist would argue that they would be arbitraged away quickly, changing the market statistics in the process. If true, then this would make the market even more effectively Markovian. However, not all traders tell others what they are doing [21].

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