Semiclassical quantization of non-Hermitian multidimensional systems using Hamilton–Jacobi equation[1]

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Abstract: The Hamilton–Jacobi equation of motion is solved in action variables for non-Hermitian systems. Both real and complex semiclassical eigenvalues are obtained that make action variables into integers. This study shows, regardless of the existence of periodic or quasi-periodic classical trajectories, Hamilton–Jacobi methods can be applied to quantize some complex non-Hermitian systems with a good accuracy.

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Résumé : Nous solutionnons les équations de Hamilton–Jacobi en fonction des variables d'action pour des systèmes non hermitiques. Nous obtenons les valeurs propres semi-classiques réelles et complexes donnant des valeurs entières aux variables d'action. Cette étude montre que, sans regard à l'existence de trajectoires périodiques ou quasi-périodiques, la méthode de Hamilton–Jacobi peut être utilisée pour quantifier certains systèmes complexes non hermitiques avec une bonne précision.

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1. Introduction

Semiclassical investigations of non-Hermitian complex Hamiltonian systems are important from both a fundamental and a practical point of view. In the first place, they expose important clues hidden in the correspondence between classical and quantum mechanics. Secondly, they provide powerful tools for the calculations of highly excited energy spectra of multi-dimensional complex systems. Since the classical trajectories of complex Hamiltonians generally lie in complex phase space, one needs to deal with complex versions of real dynamical quantities that are used in the classical mechanics of real Hermitian systems. In this study, we assume that the Hamiltonian and the canonical variables are complex quantities. Use of complex classical quantities in semiclassical quantization methods for real Hermitian systems is not uncommon. One-dimensional quantum action variable theory [1–3], semiquantum action variable theory [4], and semiclassical quantization with complex trajectories [5] are a few examples. With these methods, real eigenvalues of both Hermitian and non-Hermitian systems can be found.

In this paper, we study both periodic and nonperiodic complex systems that are non-Hermitian.

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All the trajectories of some of these systems are neither periodic nor quasi-periodic and hence action variables cannot be defined directly in the usual way, such as $\oint p\,\mathrm{d}x$ for one-dimensional theory and $\int_C p_x\,\mathrm{d}x$ and $\int_C p_y\,\mathrm{d}y$ for two-dimensional theory. In the present study, we apply the Hamilton–Jacobi methods that were developed by Capman et al. [6] for real Hermitian systems to non-Hermitian systems by analytical continuation of the formulas, and we examine the validity of the method.

The outline of the paper is as follows. In Sect. 2, we describe the Hamilton–Jacobi method developed by Capman et al. [6] and Born [7]. In Sect. 3, the formulas derived in Sect. 2 are applied to non-Hermitian one-dimensional systems by analytical continuation. We apply the Hamilton–Jacobi method to two-dimensional nonseparable non-Hermitian systems and test the validity of the method in Sect. 3. Concluding remarks are given in Sect. 4.

2. The Hamilton–Jacobi method

In this section, we briefly describe the Hamilton–Jacobi method developed for real Hermitian non-separable systems by Capman et al. [6]. Please refer to ref. 6 for details. Consider the multi-dimensional Hamiltonian of the form

$$H(\boldsymbol{p},\boldsymbol{x}) = \frac{\boldsymbol{p}^2}{2m} + V_0(\boldsymbol{x}) + V(\boldsymbol{x}) \tag{1}$$

where $(\boldsymbol{x},\boldsymbol{p}) = \{x_i, p_i\}$, $i = 1, ...f$ are the Cartesian coordinates and momenta, and f is the number of degrees of freedom. We assume that the reference potential is separable.

$$V_0(\boldsymbol{x}) = \sum_{i=1}^{f} v_i(x_i) \tag{2}$$

and we further assume that the $v_i(x_i)$ are of the form

$$v_i(x_i) = \frac{1}{2} m\omega_i^2 x_i^2 \tag{3}$$

where ω_i is a constant. The potential $V(\boldsymbol{x})$ is nonseparable and hence the problem may not have exact analytical solutions. Now we make the transformation from $(\boldsymbol{x},\boldsymbol{p})$ coordinates to action angle variables $(\boldsymbol{n},\boldsymbol{q})$ that correspond to the reference potential $V_0(\boldsymbol{x})$. Since $V_0(\boldsymbol{x})$ has the form given in (2) and (3), x_i and p_i are given by

$$x_i\,(n_i, q_i) = \left[\frac{(2n_i+1)}{m\omega_i}\right]^{1/2} \cos q_i \tag{4}$$

$$p_i\,(n_i, q_i) = -\left[(2n_i+1)\,m\omega_i\right]^{1/2} \sin q_i \tag{5}$$

The reason for having $2n_i + 1$ instead of $2n_i$ in the above formulae is that $1/2$ has to be introduced with quantum number n_i to incorporate the phase contributions from the classical turning points in the semiclassical quantization of one-dimensional systems. In terms of the action angle variables $(\boldsymbol{n},\boldsymbol{q})$, the Hamiltonian in (1) becomes

$$H(\boldsymbol{n},\boldsymbol{q}) = H_0(\boldsymbol{n}) + V(\boldsymbol{n},\boldsymbol{q}) \tag{6}$$

and

$$H_0(\boldsymbol{n}) = \sum_{i=1}^{f} \omega_i \left(n_i + \frac{1}{2}\right) = \boldsymbol{\omega}.\left(\boldsymbol{n} + \frac{1}{2}\right) \tag{7}$$

where here we have taken the units such that $\hbar = 1$. $V(\boldsymbol{n}, \boldsymbol{q})$ is written using (4) as

$$V(\boldsymbol{n}, \boldsymbol{q}) = V(\boldsymbol{x}(\boldsymbol{n}, \boldsymbol{q})) \tag{8}$$

Since $\mathrm{d}n_i/\mathrm{d}t = -\partial H_0/\partial q_i = 0$, action variables n_i are constants of motion for the Hamiltonian $H_0(\boldsymbol{n})$. Hence, when $V(\boldsymbol{n}, \boldsymbol{q}) = 0$, the eigenvalues of the system could be determined by the condition

$$E(\boldsymbol{n}) = H_0(\boldsymbol{n}) \tag{9}$$

where now $\boldsymbol{n} = \{n_i\}$, $i = 1, \ldots f$ and the n_i are integers. When $V(\boldsymbol{n}, \boldsymbol{q}) \neq 0$, the n_i are not constants of motion of the Hamiltonian $H(\boldsymbol{n}, \boldsymbol{q})$. Now we transform $(\boldsymbol{n}, \boldsymbol{q})$ to new action variables $(\boldsymbol{N}, \boldsymbol{Q})$ such that the total Hamiltonian H will depend only on the action variables $\boldsymbol{N}$. Eigenvalues of the nonseparable Hamiltonian H are given by

$$E(\boldsymbol{N}) = H(\boldsymbol{N}) \tag{10}$$

where $\boldsymbol{N} \equiv \{N_i\}$, $i = 1, \ldots f$. The existence of such action variables $\boldsymbol{N}$ for a given multi-dimensional system is assumed. If the system is quasiperiodic, then the action variables N_i can be defined as $\int_{C_i} p_i \, \mathrm{d}x_i$ where the C_i are topologically independent paths in the invariant torus. If such $\boldsymbol{N}$ can be found then the N_i become constants of motion. Further, the Hamiltonian systems we investigated are time independent and hence total energy E is also a constant of motion. Therefore, all the action variables N_i are not independent and they are related through the relation $E(\boldsymbol{N}) = E$.

If the generating function F of the above transformation $[\boldsymbol{n}, \boldsymbol{q} \rightarrow (\boldsymbol{N}, \boldsymbol{Q})]$ is assumed to be a function of $\boldsymbol{N}$ and $\boldsymbol{q}$ only then F must be of the F_2 type [7,8]. Hence, the action variables $\boldsymbol{n}$, $\boldsymbol{q}$, $\boldsymbol{N}$, and $\boldsymbol{Q}$ should satisfy the differential equations

$$n_i(\boldsymbol{N}, \boldsymbol{q}) = \frac{\partial F(\boldsymbol{N}, \boldsymbol{q})}{\partial q_i} \tag{11}$$

$$Q_i(\boldsymbol{N}, \boldsymbol{q}) = \frac{\partial F(\boldsymbol{N}, \boldsymbol{q})}{\partial N_i} \tag{12}$$

for $i = 1, \ldots f$

Combining (6) and the condition (10) with (11) we have

$$E(\boldsymbol{N}) = H_0\left(\frac{\partial F(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}}\right) + V\left(\frac{\partial F(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{\theta}}, \boldsymbol{q}\right) \tag{13}$$

This is the Hamilton–Jacobi equation for the generating function F. When $V(\boldsymbol{n}, \boldsymbol{q}) = 0$, by observing the differential (11) and (12), we write

$$\lim_{V \to 0} F(\boldsymbol{N}, \boldsymbol{q}) = \boldsymbol{q} \cdot \boldsymbol{N} \tag{14}$$

(As $V \rightarrow 0$, $(\boldsymbol{n}, \boldsymbol{q}) = (\boldsymbol{N}, \boldsymbol{Q})$)
When $V(\boldsymbol{N}, \boldsymbol{q}) \neq 0$, it is convenient to write $F(\boldsymbol{N}, \boldsymbol{q})$ as

$$F(\boldsymbol{N}, \boldsymbol{q}) = \boldsymbol{q} \cdot \boldsymbol{N} + \Omega(\boldsymbol{N}, \boldsymbol{q}) \tag{15}$$

where the function $\Omega(\boldsymbol{N}, \boldsymbol{q})$ is to be determined. Differentiating (15) and substituting in (13), we obtain the following differential equation for $\Omega(\boldsymbol{N}, \boldsymbol{q})$:

$$E(\boldsymbol{N}) = H_0\left[\boldsymbol{N} + \frac{\partial \Omega(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}}\right] + V\left[\boldsymbol{N} + \frac{\partial \Omega(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}}, \boldsymbol{q}\right] \tag{16}$$

For the present case,

$$H_0(\boldsymbol{N} + \frac{\partial \Omega(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}}) = \boldsymbol{\omega} \cdot \boldsymbol{N} + \boldsymbol{\omega} \cdot \frac{\partial \Omega(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}}$$

see, (7). Therefore,

$$E(\boldsymbol{N}) = \boldsymbol{\omega} \cdot \left(\boldsymbol{N} + \frac{1}{2}\right) + \boldsymbol{\omega} \cdot \frac{\partial \Omega(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}} + V\left[\boldsymbol{q}, \boldsymbol{N} + \frac{\partial \Omega(\boldsymbol{N}, \boldsymbol{q})}{\partial \boldsymbol{q}}\right] \tag{17}$$

The condition that the semiclassical wave function has to be single valued, implies that $\Omega(\boldsymbol{N}, \boldsymbol{q})$ must be a periodic function of q_i [9] and it can be expanded in a Fourier series as

$$\Omega(\boldsymbol{N}, \boldsymbol{q}) = i \sum_{\boldsymbol{k}}{}' B_{\boldsymbol{k}}\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{q}} \tag{18}$$

where prime on the summation implies that $\boldsymbol{k} = \boldsymbol{0}$ is omitted. Combining (11), (15), and (18), we find the relationship between $\boldsymbol{n}$ and $\boldsymbol{N}$ to be

$$\boldsymbol{n} = \boldsymbol{N} - \sum_{\boldsymbol{k}'} \boldsymbol{k}'\, \mathrm{e}^{i\boldsymbol{k}'\boldsymbol{q}} B_{\boldsymbol{k}'} \tag{19}$$

Substituting (18) into (17), multiplying by $\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{q}}$ and integrating over $\boldsymbol{q}$ yields

$$E(\boldsymbol{N})\delta_{\boldsymbol{k},0} = \boldsymbol{\omega}\left(\boldsymbol{N} + \frac{1}{2}\right)\delta_{\boldsymbol{k},0} - \boldsymbol{\omega}\cdot\boldsymbol{k}B_{\boldsymbol{k}} + \frac{1}{(2\pi)^f}\int_0^{2\pi} \mathrm{d}q\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{q}} V\left(\boldsymbol{N} - \sum_{\boldsymbol{k}'}{}' \boldsymbol{k}'\, \mathrm{e}^{i\boldsymbol{k}'\cdot\boldsymbol{q}} B_{\boldsymbol{k}'}, \boldsymbol{q}\right) \tag{20}$$

This equation can be written as

$$E(\boldsymbol{N}) = \boldsymbol{\omega}\cdot\left(\boldsymbol{N} + \frac{1}{2}\right) + \frac{1}{(2\pi)^f}\int_0^{2\pi} \mathrm{d}\boldsymbol{q}\, V\left(\boldsymbol{N} - \sum_{\boldsymbol{k}'} \boldsymbol{k}'\, \mathrm{e}^{i\boldsymbol{k}'\cdot\boldsymbol{q}} B_{\boldsymbol{k}'}, \boldsymbol{q}\right) \tag{21}$$

when $\boldsymbol{k} = \boldsymbol{0}$ and

$$\boldsymbol{\omega}\cdot\boldsymbol{k}B_{\boldsymbol{k}} = \frac{1}{(2\pi)^f}\int_0^{2\pi} \mathrm{d}\boldsymbol{q}\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{q}} V\left(\boldsymbol{N} - \sum_{\boldsymbol{k}'}{}' \boldsymbol{k}'\, \mathrm{e}^{i\boldsymbol{k}'\cdot\boldsymbol{q}} B_{\boldsymbol{k}'}, \boldsymbol{q}\right) \tag{22}$$

when $\boldsymbol{k} \neq \boldsymbol{0}$.

Now we define $\boldsymbol{A}_{\boldsymbol{k}}$ as $\boldsymbol{A}_{\boldsymbol{k}} = \boldsymbol{\omega}\cdot\boldsymbol{k}\ \boldsymbol{B}_{\boldsymbol{k}}$. Then we can write (21) and (22) as

$$E(\boldsymbol{N}) = \boldsymbol{\omega}\cdot\left(\boldsymbol{N} + \frac{1}{2}\right) + A_{\boldsymbol{0}} \tag{23a}$$

$$\boldsymbol{A}_{\boldsymbol{k}} = \frac{1}{(2\pi)^f}\int_0^{2\pi} \mathrm{d}\boldsymbol{q}\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{q}} V\left(N - \sum_{\boldsymbol{k}'} \frac{\boldsymbol{k}'\, \mathrm{e}^{i\boldsymbol{k}'\cdot\boldsymbol{q}}}{\boldsymbol{\omega}\cdot\boldsymbol{k}'}\boldsymbol{A}_{\boldsymbol{k}'}, \boldsymbol{q}\right) \tag{23b}$$

By using (4) and (19), (23b) may be written as

$$\boldsymbol{A}_{\boldsymbol{k}} = \frac{1}{(2\pi)^f}\int_0^{2\pi} \mathrm{d}\boldsymbol{q}\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{q}} V(\boldsymbol{x}(\boldsymbol{q})) \tag{24}$$

where

$$x_i(\boldsymbol{q}) = \left[\frac{(2n_i+1)}{m\omega_i}\right]^{1/2} \cos q_i \tag{25a}$$

and

$$n_i = N_i - \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}_i \, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{q}}}{\boldsymbol{\omega}\cdot\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}} \tag{25b}$$

for $i = 1, 2, 3.....f$. In this method, (24) and (25) are solved iteratively to find A_0 and then (23a) is used to find semiclassical eigenenergies $E(\boldsymbol{N})$ for given integers $\boldsymbol{N} \equiv \{N_i\}$. The above(23)–(25) are derived for the real Hermitian Hamiltonians having "good" action variables. Therefore, these equations are not valid when the classical phase space is completely chaotic.

3. One-dimensional non-Hermitian systems

In the one-dimensional time independent systems, which we consider in this section, the Hamiltonian is a constant of motion and equal to the total energy of the system. The first one-dimensionalsystem of interest is

$$H_1 = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 + igx^3 \tag{26}$$

where ω and g are real constants and the Hamiltonian H_1 is PT symmetric. The quantum energy spectrum of this system is entirely real [10]. Before applying the Hamilton–Jacobi method described in the previous section, we study the classical motion of this system in complex phase space. A typical classical trajectories of H_1 is shown in Fig. 1*a*. It is periodic and the action variable can be defined by $J = \oint p \, \mathrm{d}x$ where the path of integration is taken over a complete cycle in complex phase space. The one-dimensional version of (23)–(25) are

$$E(N) = \omega\left(N + \frac{1}{2}\right) + A_0 \tag{27}$$

and

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}q \, \mathrm{e}^{-ikq} V(x(q)) \tag{28}$$

where

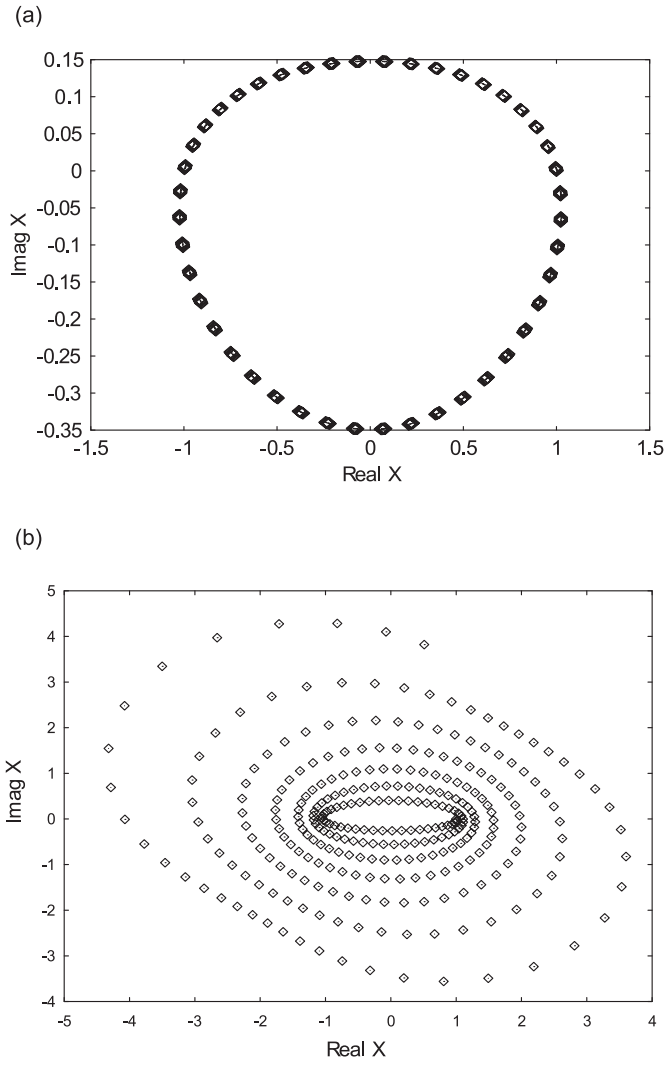
$$x(q) = \left[\frac{(2n+1)}{m\omega}\right]^{1/2} \cos q \tag{29a}$$

and

$$n = N - \frac{1}{\omega}\sum_{k'} \mathrm{e}^{ik'q} A_{k'} \tag{29b}$$

For Hamiltonian H_1, we take mass $m = 1.0$, $\omega = 1.0$, and $g = 0.01$. Then solve (28) and (29) iteratively until A_0 converges. Then (27) is used to calculate the semiclassical eigenvalues. For comparison purposes quantum eigenenergies are found by diagonalizing the Hamiltonian H_1 in the one-dimensional harmonic oscillator basis set. The results are presented in Table 1.

Fig. 1. (*a*) A typical classical trajectory of the complex PT symmetric Hamiltonian $H_1 = p^2/2m + 1/2\omega^2x^2 + igx^3$ in complex x plane. This trajectory is periodic. (*b*) A typical classical trajectory of the complex non-pseudo Hermitian Hamiltonian $H_2 = p^2/2 + 1/2\omega^2x^2 + igx^4$ in complex x plane. This trajectory is nonperiodic and never closes itself.



Although the potential $V(x)$ is complex, the semiclassical Fourier coefficients A_0 were found to be real for all N. The semiclassical eigenenergies obtained with the Hamilton–Jacobi method are in excellent agreement with the exact energies.

The second potential we investigate in this section is $H_2 = \frac{p^2}{2} + \frac{1}{2}\omega^2x^2 + igx^4$. This complex Hamiltonian H_2 is not pseudo Hermitian and all the quantum eigenvalues are complex. A typical classical trajectories of H_2 is shown in Fig. 1*b*.

It is evident from Fig. 1*b* that the trajectory is nonperiodic and never closes itself. It spirals around and the particle escapes to complex infinity. Therefore, the action variable cannot be defined as in the previous case. However, we apply (27)–(29) to this system and estimate the semiclassical eigenenergies by analytically continuing the parameters in the above equations to the complex plane. Unlike the previous system, the Fourier coefficients A_0 are found to be complex. The results are shown in Table 2.

Table 1. First ten eigenenergies of $V(x) = \frac{1}{2}\omega^2 x^2 + igx^3$ where $\omega = 1.0$ and $g = 0.01$. Exact eigenenergies are obtained by diagonalizing the Hamiltonian in the harmonic oscillator basis set.

N	Exact energy	Semiclassical energy
0	0.500 14	0.500 094
1	1.500 89	1.500 84
2	2.502 38	2.502 34
3	3.504 62	3.504 57
4	4.507 59	4.507 55
5	5.511 31	5.511 27
6	6.515 77	6.515 72
7	7.520 95	7.520 91
8	8.526 87	8.526 83
9	9.533 51	9.533 47

Table 2. First ten eigenenergies of $V(x) = \frac{1}{2}\omega^2 x^2 + igx^4$ where $\omega = 1.0$ and $g = 0.01$. Exact eigenenergies are obtained by diagonalizing the Hamiltonian in the harmonic oscillator basis set.

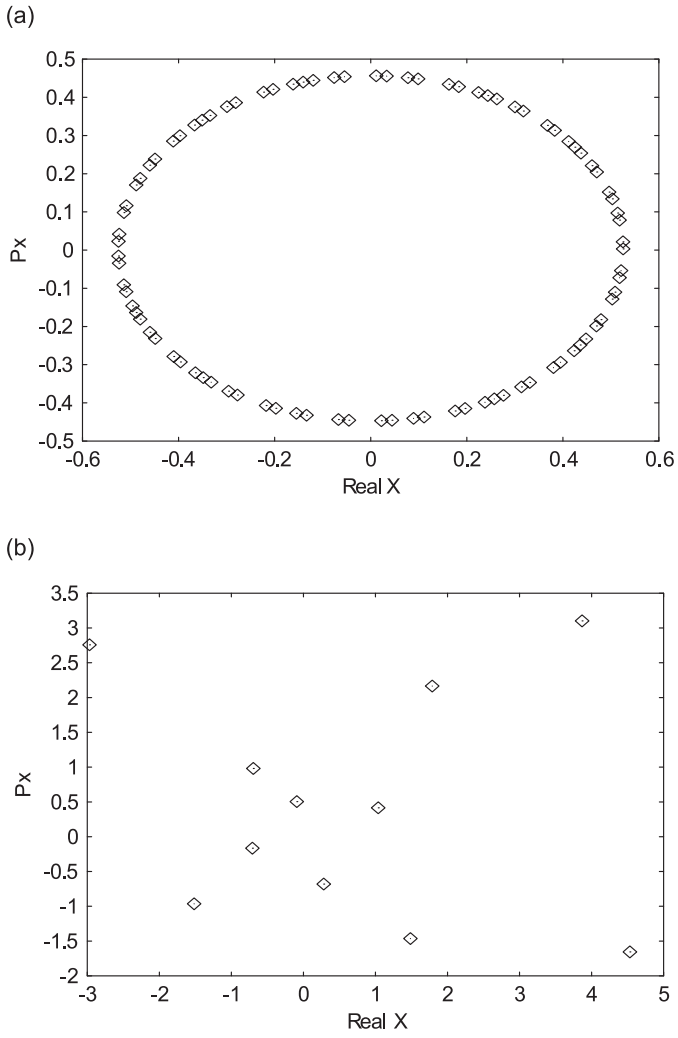
N	Exact energy	Semiclassical energy
0	$0.5002 + i0.0074$	$0.5001 + i0.0038$
1	$1.5020 + i0.0373$	$1.5014 + i0.0336$
2	$2.5074 + i0.0963$	$2.5066 + i0.0928$
3	$3.5186 + i0.1837$	$3.5174 + i0.1804$
4	$4.5374 + i0.2981$	$4.5360 + i0.2952$
5	$5.5653 + i0.4380$	$5.5639 + i0.4354$
6	$6.6033 + i0.6021$	$6.6017 + i0.5994$
7	$7.6522 + i0.7886$	$7.6502 + i0.7865$
8	$8.7125 + i0.9964$	$8.7111 + i0.9942$
9	$9.7845 + i1.2239$	$9.7827 + i1.2224$

It is evident from the Table 2 that the agreement between the eigenvalues obtained using the Hamilton–Jacobi method and the exact eigenenergies are in good agreement although the action variable cannot be defined as in the previous case. This indicates that even for non-pseudo Hermitian complex systems having nonperiodic trajectories, the action variable may exist. However, further studies are needed to define it properly.

4. The two-dimensional non-Hermitian systems

Classical trajectories in the phase space of multi-dimensional Hermitian systems are usually categorized as either regular or irregular. For a real two-dimensional potential, regular trajectories occupy a limited amount of the energetically allowed phase space and have well-defined Poincare' surfaces of section. In this case, there exists a constant of motion in addition to the energy and it is the action variable. On the other hand, irregular trajectories appear to fill up the allowed phase space and have effectively random Poincare' surfaces [11] due to the absence of this additional constant of motion. In this case, the action variable cannot be defined. For two-dimensional Hamiltonian systems, the Hamilton–Jacobi method described in Sect. 2 is valid only when there exists a constant of motion N, other than the total energy. To identify the regions of classical phase space where such constants of motion exist, the

Fig. 2. (*a*) Typical Poincare surfaces of section (or projections) for the complex PT-symmetric potential Hamiltonian $H_1 = p_x^2/2 + p_y^2/2 + w_x^2x^2 + w_y^2y^2 + igxy^2$ with $w_x = 0.7$, $w_y = 1.3$, and $g = 0.1$. This represents a regular quasi-periodic classical motion. (*b*) Typical Poincare surfaces of section (or projections) for the complex non-Hermitian Hamiltonian $H_4 = p_x^2/2 + p_y^2/2 + w_x^2x^2 + w_y^2y^2 + igx^2y^2$ with $w_x = 0.7$, $w_y = 1.3$, $g = 0.06$, and $E = 1.0$. This represents a non-quasi-periodic classical trajectory.



classical trajectories have to be investigated to find whether they are either regular or chaotic (irregular). For both Hermitian and non-Hermitian systems, Poincare' surfaces of section are normally used to distinguish regular trajectories from irregular ones [12]. As in the previous section, before we apply the Hamilton–Jacobi method for two-dimensional non-Hermitian systems, we investigate the classical phase space.

The first two-dimensional system we study is the PT-symmetric complex Barbanis potential $V(x, y) = w_x^2x^2+w_y^2y^2+igxy^2$, which is known to have all eigenvalues real. For the values $w_x = 0.7, w_y = 1.3$, and $g = 0.1$, most of the classical trajectories of this system are quasi-periodic and the Poincare sections are found to be closed curves when the energy is below 15.0. as shown in Fig. 2*a*.

Since a constant of motion other than the total energy exists when the motion is regular, action variable N can be defined. Therefore, the Hamilton–Jacobi method discussed in Sect. 2 can be applied.

Table 3. First five eigenenergies of $V(x, y) = w_x^2x^2 + w_y^2y^2 + igxy^2$ with $w_x = 0.7$, $w_y = 1.3$, and $g = 0.1$. Exact eigenenergies are obtained by diagonalizing the Hamiltonian in the harmonic oscillator basis set.

N1	N2	Exact energy	Semiclassical energy
0	0	1.0021	1.0022
1	0	1.7038	1.7033
0	1	2.3152	2.3171
2	0	2.4055	2.4044
1	1	3.0202	3.0202

For two-dimensional systems, (23)–(25) become

$$A_{k_1,k_2} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{e}^{-i(k_1q_1+k_2q_2)}V(x_1(q_1, q_2), x_2(q_1, q_2)) \tag{30}$$

where

$$x_1(q_1, q_2) = \left[\frac{(2n_1(q_1, q_2) + 1)}{\omega_1}\right]^{1/2}\cos q_1 \tag{31a}$$

$$x_2(q_1, q_2) = \left[\frac{(2n_2(q_1, q_2) + 1)}{\omega_2}\right]^{1/2}\cos q_2 \tag{31b}$$

$$n_1(q_1, q_2) = N_1 - {\sum_{k_1',k_2'}}' \frac{k_1\,\mathrm{e}^{i(\omega_1k_1+\omega_2k_2)}}{\omega_1k_1 + \omega_2k_2}A_{k_1',k_2'} \tag{32a}$$

and

$$n_2(q_1, q_2) = N_2 - {\sum_{k_1',k_2'}}' \frac{k_2\,\mathrm{e}^{i(\omega_1k_1+\omega_2k_2)}}{\omega_1k_1 + \omega_2k_2}A_{k_1',k_2'} \tag{32b}$$

$$E(N_1, N_2) = \omega_1\left(N_1 + \frac{1}{2}\right) + \omega_2\left(N_2 + \frac{1}{2}\right) + A_0 \tag{33}$$

We solve (30)–(32) iteratively and obtain A_0. Then (33) is used to obtain semiclassical energies. Table 3 shows the first five eigenenergies calculated with the above method. The agreement between the eigenvalues obtained using the Hamilton–Jacobi method and the exact method is clearly evident from Table 3.

The second system we investigate is the non-Hermitian system $V(x, y) = w_x^2x^2 + w_y^2y^2 + igx^2y^2$ for which all the eigenvalues are complex. Unless the energy is very small ($E \ll 1.0$), the classical trajectories are found to be non quasi-periodic and the Poincare surfaces of sections are not closed curves as shown in Fig. 2*b*.

Since the Poincare surfaces of sections are not closed curves, but randomly distributed points, the action variable N cannot be defined as in the previous case. However, we apply the formulas given in

Table 4. First five eigenenergies of $V(x, y) = w_x^2x^2 + w_y^2y^2 + igx^2y^2$ with $w_x = 0.7$, $w_y = 1.3$, and $g = 0.06$. Exact eigenenergies are obtained by diagonalizing the Hamiltonian in the harmonic oscillator basis.

N1	N2	Exact energy	Semiclassical energy
0	0	$1.0008 + i0.0164$	$1.0007 + i0.0164$
1	0	$1.7037 + i0.0488$	$1.7037 + i0.0489$
0	1	$2.3047 + i0.0485$	$2.3047 + i0.0485$
2	0	$2.4090 + i0.0803$	$2.4085 + i0.0805$
1	1	$3.0171 + i0.1440$	$3.0179 + i0.1437$

(30)–(33) to this system and obtain the semiclassical energies. In Table 2 the semiclassical energies are compared with the exact energies.

It is evident that the semiclassical energies are in good agreement with the exact energies although the action variable cannot be defined as in the previous case.

5. Concluding remarks

In this investigation, we studied both one-dimensional and two-dimensional non-Hermitian systems semiclassically. For one-dimensional real Hermitian systems, the existence of periodic trajectories is important for semiclassical quantization while for multi-dimensional systems, existence of invariant tori is required. In other words, periodic or quasiperiodic motion is required for semiclassical quantization. The Hamilton–Jacobi method developed by Capman et al. [6] and Born [7], which is the main subject of this paper, assumes the existence of the invariant tori and hence the integrability of the system. With two examples of one-dimensional non-Hermitian systems, we showed that even when a system is not having periodic trajectories in complex phase space, the Hamilton–Jacobi method produced accurate semiclassical eigen values. Further, for multi-dimensional systems, even when the Poincare surfaces of section have random points, we showed with two examples that the Hamilton–Jacobi method produced accurate semiclassical eigenvalues. In other words, nonperiodic and nonquasi-periodic trajectories in non-Hermitian systems some times behave as periodic in some sense. Therefore, the way the action variables defined for real Hermitian systems may have to be changed for non-Hermitian systems.

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