# Asymptotic behavior of eigen energies of non-Hermitian cubic polynomial systems<sup>1</sup>

# Asiri Nanayakkara

Abstract: The asymptotic behavior of the eigenvalues of a non-Hermitian cubic polynomial system  $H = (P^2/2) + \mu x^3 + ax^2 + bx$ , where  $\mu$ , a, and b are constant parameters that can be either real or complex, is studied by extending the asymptotic energy expansion method, which has been developed for even degree polynomial systems. Both the complex and the real eigenvalues of the above system are obtained using the asymptotic energy expansion. Quantum eigen energies obtained by the above method are found to be in excellent agreement with the exact eigenvalues. Using the asymptotic energy expansion, analytic expressions for both level spacing distribution and the density of states are derived for the above cubic system. When  $\mu = i$ , a is real, and b is pure imaginary, it was found that asymptotic energy level spacing increases with the coupling strength a for positive a while it decreases for negative a.

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**Résumé :** À l'aide d'une extension de la méthode d'expansion asymptotique en énergie qui a été développée pour des potentiels en polynôme de degré pair, nous étudions le comportement asymptotique des valeurs propres de systèmes à potentiel polynomial cubique non hermitique  $H = (P^2/2) + \mu x^3 + ax^2 + bx$ , où  $\mu$ , *a* et *b* sont des constantes nues qui peuvent être réelles ou complexes. Nous obtenons de cette expansion asymptotique les valeurs propres réelles et imaginaires. Nous observons que les valeurs propres quantiques obtenues ci-dessus sont en excellent accord avec les valeurs propres exactes. L'utilisation de l'expansion asymptotique nous donne des expressions analytiques pour la distribution de la séparation entre les niveaux et la densité d'états pour le système cubique étudié ici. Lorsque nous prenons  $\mu = i$ , *a* réel et *b* purement imaginaire, nous trouvons que la séparation asymptotique en énergie entre les niveaux augmente avec la force de couplage pour *a* positif et décroît pour *a* négatif.

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### 1. Introduction

Many non-Hermitian PT symmetric quantum Hamiltonians are found to have real spectra when the PT symmetry is not spontaneously broken [1, 2]. Although the real cubic oscillator potentials have no

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**A. Nanayakkara.** Institute of Fundamental Studies, Hanthana Road, Kandy, Sri Lanka (e-mail: asiri@uwu.ac.lk).

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true bound states, the PT symmetric version of it,  $H = (P^2/2) + ix^3$ , is known to have infinitely many bound states and the entire quantum spectrum of this system is real and positive [1, 3–5]. On the other hand, when the PT symmetry of a PT symmetric non-Hermitian Hamiltonian is spontaneously broken or a non-Hermitian Hamiltonian becomes nonpseudo-Hermitian, the eigen spectrum of such a system becomes complex.

General cubic PT symmetric systems have been studied intensively in recent past [5–8]. Specifically, Znojil et al. have investigated the asymptotic solvability of a cubic oscillator with spikes [5] and Fernandez et al. have studied the convergence of the perturbation expansion of the cubic polynomial system with numerical perturbation theory [8]. However, the asymptotic behavior of the eigenvalues of a cubic non-Hermitian polynomial system has not been studied so far.

The asymptotic energy expansion (AEE) method [9] is useful in obtaining analytic expressions for the quantum action variable J(E) as a power series of energy E. One of the interesting features of the AEE method is its ability to provide analytic expressions for coefficients that are present in the exact resolution method of Voros [10] as unknown coefficients. When  $V(x) = x^{2N}$ , the AEE method and the higher order Wentzel-Kramers-Brillouin (WKB) method [11] produce the same expansion and hence the AEE method also produces the exact eigen energy for the simple harmonic oscillator potential (i.e., when N = 1). However, when V(x) contains additional terms (for example,  $V(x) = x^{2N} + bx^2$ ), terms in the high-order WKB and AEE methods are entirely different. The WKB expansion can be considered as a rearrangement of the terms of the AEE as a power series in  $\hbar$ . Nevertheless, as both series are infinite series, the rearrangement of terms is a nontrivial task, especially, for the polynomial potentials such as the cubic non-Hermitian system studied here, higher order AEE terms can be derived algebraically, even though, higher order WKB terms cannot be derived as algebraic expressions owing to the complexity of integrals involved.

The analytic form of the AEE formulae is not only important in providing a method for obtaining accurate quantum eigenvalues but also it is a valuable tool in providing analytic insight into the system. The parameters in the polynomial potentials are present in the AEE explicitly. Therefore, the AEE is ideally suited for studying the asymptotic behavior of the eigen energies of polynomial systems with respect to the changes in the coefficients of the potential. So far, only even-degree polynomial potentials have been studied using the AEE [9, 12–16]. For PT symmetric polynomial potentials, asymptotic expressions for the eigen energies have been derived by Sibuya [17] and Shin [18] and the WKB expressions of energy *E* and *C* operators have been derived for cubic potentials by Bender et al. [1, 19]. However, the AEE derived here is not an asymptotic expansion of energy *E* in some parameter, but an asymptotic expansion of the action variable J(E) for large *E* and J(E) becomes  $n\hbar$  (*n* is an integer) for eigen energies.

The purpose of this paper is twofold. First it is to extend the AEE method for odd-degree cubic PT symmetric polynomial systems and investigate how accurate it is in predicting eigen values of odd-degree non-Hermitian polynomial potentials. Second is to study analytically, how the asymptotic level spacings of adjacent states and the density of states change with parameters in the cubic polynomial system.

# 2. Asymptotic energy expansion of cubic general complex polynomial potential

Now consider the general cubic non-Hermitian Hamiltonian system

$$H = \frac{P^2}{2} + \mu x^3 + ax^2 + bx \tag{1}$$

where  $\mu$ , a, and b are either real or complex and the system can be Hermitian, PT symmetric, or nonpseudo Hermitian according to the parameters  $\mu$ , a, and b. Table 1 shows the Hermiticity of this system for various values of  $\mu$ , a, and b.

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Table 1. The system given in (1) can be Hermitian, PT symmetric, or non-pseudo Hermitian. When all the parameters are real, the system is Hermitian. However, in this case, there are no true quantum bound states. When the system is PT symmetric, quantum eigenvalues are found to be real and positive. If any of these three parameters are complex, the system is non-pseudo Hermitian and quantum eigenvalues are complex.

i	a	b	System
Real	Real	Real	Hermitian
Real	Real	Imaginary	Non-Hermitian and PT-symmetric
Imaginary	Real	Imaginary	Non-Hermitian and PT symmetric
Imaginary	Real	Real	Non-pseudo Hermitian
Real or Complex	Real or Complex	Real or Complex	Non-pseudo Hermitian (if all are not real)

In this paper, we derive a single analytic expression for quantum action J(E), which can be used to calculate quantum eigenvalues of system (1) for all the cases in Table 1.

The Schrödinger equation for the system in (1) is given by

$$-\hbar^2 \frac{\partial^2 U(x, E)}{\partial x^2} + V(x)U(x, E) = EU(x, E)$$
<sup>(2)</sup>

where  $V(x) = \mu x^3 + ax^2 + bx$ . Substituting

$$P(x, E) = \frac{\hbar}{i} \frac{\partial U(x, E) / \partial x}{U(x, E)}$$

in (1), we obtain the Riccati equation

$$\frac{\hbar}{i}\frac{\partial P(x,E)}{\partial x} + P^2(x,E) = E - \mu x^3 - ax^2 - bx$$
(3)

For the *N*th degree polynomial system  $V(x) = x^N + a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + \dots + a_1x$ , the AEE method has been derived for even *N* by the transformation  $E = 1/\varepsilon^N$  and  $y = \mu^{1/N} \varepsilon x$  [13]. For even N, after this transformation, (3) will contain terms with integer powers of  $\epsilon$ . However, when N is odd, under the same transformation, (3) will have terms with nonintegers of  $\epsilon$ . By re-examining the AEE method, it was found that if we let  $E = 1/\varepsilon^6$  and  $y = \mu^{1/3} \varepsilon^2 x$ , (3) can be brought to a form where asymptotic expansion can be derived.

$$\widetilde{h}\varepsilon^8 \frac{\partial p(y, E)}{\partial y} + \varepsilon^6 P^2(y, E) = 1 - y^3 - A y^2 \varepsilon^2 - B y \varepsilon^4$$
(4)

where  $\tilde{h} = \frac{\hbar}{i} \mu^{1/3}$ ,  $A = \frac{a}{\mu^{2/3}}$ , and  $B = \frac{b}{\mu^{1/3}}$ . Let  $P(y, E) = \varepsilon^s \sum_{k=0}^{s} a_k(y) \varepsilon^k$ . Substituting P(y, E) in (4) and equating coefficients for powers of  $\varepsilon$ , we obtain recurrence relations for  $a_k$ .

$$a_0 = \sqrt{1 - y^3} \tag{5a}$$

 $a_1 = 0$ 

s = -3 and

$$a_2 = -\frac{1}{2a_0}Ay^2$$
(5b)

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 $a_3 = 0$ 

$$a_4 = -\frac{1}{2a_0} \left( By + a_2^2 \right) \tag{5c}$$

and for  $n \ge 5$ 

$$a_{n} = -\frac{1}{2a_{0}} \left\{ \sum_{k=1}^{n-1} a_{k} a_{n-k} + \tilde{h} \frac{\mathrm{d}a_{n-5}}{\mathrm{d}y} \right\}$$
(5d)

Now we use the quantum action variable J(E), defined as

$$J(E) = \frac{1}{2\pi} \int_{C} P \,\mathrm{d}x \tag{6a}$$

where *P* is given inn (3) and *C* encloses the two physical turning points of classical momentum function  $P_C = \sqrt{E - V(x)}$ . Details of these definitions are described in ref. 20. For the system in (1), the action variable J(E) is

$$J(E) = \sum_{k=0}^{\infty} b_k E^{-(k-5)/6}$$
(6b)

where  $b'_k s$  are given by

$$b_k = \frac{1}{2\pi\tilde{h}} \oint_C a_k(y) \,\mathrm{d}y \tag{7}$$

The contour *C* in the integral (7) encloses y = 1 and  $y = \infty$ . This integral is quite different from the integrals encountered in obtaining asymptotic expressions for even degree polynomial potentials[9, 12–16]. To evaluate the integral, we make the transformation s = 1/y. Now the integral becomes

$$b_k = -\frac{1}{2\pi\tilde{h}} \oint_{C_S} s^{-2} a_k(s) \,\mathrm{d}s \tag{8}$$

The contour  $C_S$  encloses points s = 0 and s = 1 and the section of the real axis between them. The integral (8) is obtained in terms of  $\Gamma$  functions. The first seven nonzero coefficients in terms of  $\hbar$ ,  $\mu$ , a, and b are

$$b_{0} = \frac{\Gamma[1/6]}{5\sqrt{\pi}\Gamma[2/3]}$$

$$b_{4} = -\frac{(a^{2} - 3b\mu)}{\mu^{4/3}} \frac{\Gamma[5/6]}{3\sqrt{\pi}\Gamma[1/3]}$$

$$b_{5} = -\frac{\hbar}{2i}\mu^{1/3}$$

$$b_{6} = -\frac{a}{\mu^{2}} \left(9b\mu - 2a^{2}\right) \frac{\Gamma[1/6]}{162\sqrt{\pi}\Gamma[2/3]}$$
(9)

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$$b_{10} = -\frac{1}{\mu^{10/3}} (4a^5 - 30\mu a^3 b + 54\mu^2 a b^2 + 81\mu^4 \hbar^2) \frac{\Gamma[5/6]}{972\sqrt{\pi}\Gamma[1/3]}$$

$$b_{12} = -\frac{1}{\mu^4} \left(7a^6 - 63\mu a^4 b + 162\mu^2 a^2 b^2 - 81\mu^3 b^3\right) \frac{\Gamma[1/6]}{52488\sqrt{\pi}\Gamma[2/3]}$$

$$b_{16} = \frac{5}{\mu^{16/3}} \left(11a^8 - 132\mu a^6 b + 540\mu^2 a^4 b^2 - 810\mu^3 a^2 b^3 + 243\mu^4 b^4 - 324\mu^4 a^3 \hbar^2 + 1458\mu^5 a b \hbar^2\right) \frac{\Gamma[5/6]}{314928\sqrt{\pi}\Gamma[1/3]}$$

In this method the quantum eigen energies are obtained by imposing the quantization condition  $J(E) = n\hbar$ .

# 3. Asymptotic energies and level spacings

Now we show the accuracy of the above formulae with two illustrations. We calculate first ten quantum eigen energies of two cubic systems, the first system is a PT symmetric version of (1) with  $\mu = i$ , a = 0.1, and b = 0.1i and all of its eigen values are real and positive while the second system is the non pseudo-Hermitian version of (1) with  $\mu = i$ , a = 0.1, and b = 0.1 and all eigen values are complex.

Table 2 shows the eigen values of the first system, obtained by the formulae given in equations (6) and (9) and for comparison purposes exact eigen energies obtained by diagonalizing the Hamiltonian in harmonic oscillator basis sets are also presented. It is evident from Table 2 that, except for the first few lowest states, the agreement between the two methods is excellent. Since the asymptotic energy expansion method is a semiclassical method, it produces very accurate results for higher excited states. Similarly, the eigen values of the second system which is the non pseudo Hermitian version of (1) are obtained with the first seven nonzero terms in (6) and the exact eigen energies are obtained by diagonalizing the Hamiltonian in harmonic oscillator basis sets. The results are shown in Table 3. As in the PT symmetric case, the agreement between the two methods is obvious.

Now, using the AEE expansion in (6), we investigate how the asymptotic level spacings of adjacent states and the density of states change, with parameters a and b in the PT symmetric version of (1)

$$H = \frac{P^2}{2} + ix^3 + ax^2 + ibx$$
(10)

where now *a* and *b* are real numbers. Here, we have taken  $\mu = i$  and  $b \rightarrow ib$ . Hence, the Hamiltonian is PT symmetric and the eigenvalues become real and positive. Since the AEE method provides an algebraic expression for the quantum action variable J(E) in terms of energy *E* as an asymptotic series, the level spacing distribution of above system can be studied analytically. For small  $\Delta E$ 

$$\Delta J(E) = \frac{\partial J(E)}{\partial E} \Delta E \tag{11}$$

Here,  $\Delta$  refers to the difference in adjacent eigen energy levels and hence  $\Delta J = 1$  and  $\Delta E$  is the level spacing between two adjacent eigen energy levels. Therefore, the energy level spacing is given approximately by

$$\Delta E = \frac{1}{\frac{\partial J(E)}{\partial E}} \tag{12}$$

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**Table 2.** System 1 is PT symmetric with  $\mu = i$ , a = 0.1, and b = 0.1i. Eigenvalues are calculated using the first seven nonzero terms in the asymptotic energy expansion (AEE) while exact eigen energies are obtained by diagonalizing the Hamiltonian in harmonic oscillator basis sets.

п	AEE	Exact
0	1.202 850 80	1.215 096 54
1	4.208 075 85	4.207 916 40
2	7.683 949 60	7.683 945 57
3	11.454 032 38	11.454 031 35
4	15.446 224 50	15.446 224 05
5	19.619 361 15	19.619 360 88
6	23.946 364 92	23.946 364 73
7	28.407 897 85	28.407 897 70
8	32.989 375 39	32.989 375 30
9	37.679 360 36	37.679 360 46

**Table 3.** System 2 is non pseudo-Hermitian with  $\mu = i$ , a = 0.1, and b = 0.1. Eigenvalues are calculated using the first seven nonzero terms in the asymptotic energy expansion (AEE) while exact eigen energies are obtained by diagonalizing the Hamiltonian in harmonic oscillator basis sets.

n	AEE	Exact
0	1.14210138 + i0.0588627	1.15696845 + i0.0557648
1	4.11206781 + i0.0948185	4.11185576 + i0.0949094
2	$7.56574290 + i \ 0.1172398$	7.56573902 + i 0.1172459
3	$11.31852694 + i \ 0.1346612$	11.31852683 + i0.1346629
4	15.29618543 + i0.1492759	15.29618573 + i0.1492765
5	19.45661555 + i0.1620415	19.45661583 + i0.1620418
6	23.77223069 + i0.1734757	23.77223092 + i0.1734759
7	28.22338085 + i0.1838952	28.22338103 + i0.1838953
8	32.79527433 + i0.1935097	32.79527437 + i0.1935097
9	37.47632833 + i0.2024664	37.47632876 + i0.2024671

Hence, level spacing for the above system for large E is given by

$$\Delta E \simeq \frac{6\sqrt{\pi}\,\Gamma[2/3]}{\Gamma[1/6]} E^{1/6} \left\{ 1 + E^{-2/3}(a^2 + 3b) \frac{\Gamma[5/6]\,\Gamma[2/3]}{\Gamma[1/6]\,\Gamma[1/3]} \right\}$$
(13)

and for very large energies

$$\Delta E \sim n^{1/5} \tag{14}$$

where *n* is the quantum number. Therefore, level spacing varies as one fifth power of the quantum number for the highly excited states for very large energies. Now let us see how the eigen energy changes with parameters *a* and *b*. The first-order approximation to the change in energy  $\Delta E$  is given by

$$\Delta E_a = -\frac{\partial J(E)/\partial a}{\partial J(E)/\partial E} \Delta a \tag{15}$$

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and

$$\Delta E_b = -\frac{\partial J(E)/\partial b}{\partial J(E)/\partial E} \Delta b \tag{16}$$

where  $\Delta E_a$  and  $\Delta E_b$  are the changes in energy due to small changes in a and b, respectively. Hence,

$$\Delta E_a = a \left[ \frac{4 \Gamma[5/6] \Gamma[2/3]}{\Gamma[1/6] \Gamma[1/3]} \right] E^{1/3} \Delta a \tag{17}$$

and the sign of a determines how the level spacing between two adjacent states change with the parameter a. When a is positive, the level spacing between two adjacent states increases as a increases. However, when a is negative, level spacing between two adjacent states decreases as |a| increases. On the other hand

$$\Delta E_b = \left[\frac{6\,\Gamma[5/6]\Gamma[2/3]}{\Gamma[1/6]\Gamma[1/3]}\right] E^{1/3}\Delta b \tag{18}$$

and as b increases, level spacing between two adjacent states increases regardless of the sign of b.

Using the asymptotic expression J(E), the density of states can be calculated as well. The density of states d(E) is defined such that  $\int_{E_a}^{E_b} d(E) dE$  is the number of states with energy levels between  $E_a$  and  $E_b$ . Thus,  $d(E) = \sum_i \delta(E - E_i)$  where  $E_i \le E_{i+1}$ . Since  $J(E) = n\hbar$  is satisfied by all the eigenvalues (i.e., it is satisfied by the *n*th eigen state for all *n*), the density of states is given by

$$d(E) = \frac{\partial J(E)}{\partial E}$$
(19)

Hence, the density of states of the above system is calculated approximately as

$$d(E) \simeq \frac{\Gamma[1/6]}{6\sqrt{\pi}\Gamma[2/3]} E^{-1/6} \left\{ 1 + E^{-2/3}(a^2 + 3b) \frac{\Gamma[5/6]\Gamma[2/3]}{\Gamma[1/6]\Gamma[1/3]} \right\}$$
(20)

and for very large energies, the density of states varies with quantum number n as

$$d(E) \sim n^{-1/5} \tag{21}$$

# 4. Concluding remarks

In this paper, we derived an analytic energy expansion of quantum action variable J(E) for the general cubic system  $H = (P^2/2) + \mu x^3 + ax^2 + bx$  and obtained quantum eigen energies. The accuracy of the method was demonstrated for two cases where in the first case the above system is PT symmetric and the energy eigenvalues are real while in the second case the above system becomes non pseudo-Hermitian and the eigenvalues are complex. One of the attractive features of the derived formulae is that single expression (6) includes both PT symmetric and non pseudo-Hermitian cases while formulae are independent of the complicated distribution of locations of branch points of the momentum  $p = \sqrt{E - \mu x^3 + ax^2 + bx}$  when  $\mu$ , a, and b are complex. The analytic formula of level spacing shows that the sign of a determines how the level spacing between two adjacent states increases as a increases. However, when a is negative, level spacing between two adjacent states decreases as |a| increases. On the other hand, as b increases, level spacing between two adjacent states increases regardless of the sign of b.

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