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ARTICLE

Effects of complex time on periodic and nonperiodic classical trajectories of one-dimensional Hamiltonian systems

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Abstract: In recent years, much research has been carried out on extending both quantum mechanics and classical mechanics into the complex domain by making parameters of real hermitian Hamiltonians or total energy of the system complex. In this paper we investigate the effects of complex time on periodic and nonperiodic trajectories of both hermitian and nonhermitian one-dimensional classical Hamiltonian systems. Most of the periodic classical trajectories of real hermitian systems turn into nonperiodic and open when the energy or the parameters of the potential become complex. We show that when time is taken as a complex quantity with a specific fixed phase angle or as a specific complex function, nonperiodic trajectories become periodic and closed. Furthermore, we show that real hermitian systems, such as $H = p^2/2m + x^4 + bx^3 + cx^2 + dx$ (b, c, and d are real quantities) possess classical periodic trajectories for real energies even when time is complex (i.e., $t = t_r e^{i\tau}$). It was found that there is a discrete set of τ values for which the trajectories of the preceding system are closed and periodic and periods associated with them form a countably infinite set.

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Résumé : Ces récentes années, beaucoup de recherche s'est faite pour étendre les mécaniques quantique et classique vers le domaine complexe en rendant complexes les paramètres réels du Hamiltonien adjoints ou l'énergie totale du système. Nous analysons ici les effets du temps complexe sur les trajectoires périodiques et non périodiques de systèmes hamiltoniens classiques 1-D adjoints et non adjoints. La plupart des trajectoires périodiques classiques de systèmes réels adjoints deviennent non périodiques et ouvertes quand l'énergie ou les paramètres du potentiel deviennent complexes. Nous montrons que lorsque le temps est pris complexe, avec un angle de phase spécifique constant ou comme une fonction complexe spécifique, des trajectoires non périodiques deviennent fermées et périodiques. De plus, nous montrons que des systèmes hamiltoniens réels comme $H = p^2/2m + x^4 + bx^3 + cx^2 + dx$ (b, c, d, quantités réelles) possèdent des trajectoires classiques périodiques avec des énergies réelles, même lorsque le temps est de la forme complexe $t = t_r e^{i\tau}$. Nous trouvons qu'il y a un ensemble discret de valeurs de τ pour lesquelles les trajectoires du système au dessus sont fermées et périodiques et les périodes associées forment un ensemble infini dénombrable. [Traduit par la Rédaction]

1. Introduction

In many branches of physics, Hamiltonian systems that possess periodic trajectories are of especial importance. In multidimensional real Hamiltonian systems, the knowledge of periodic or quasiperiodic trajectories is valuable in understanding ergodicity and determining semiclassical eigenenergies using Einstein–Brillouin–Keller (EBK)-like quantization methods [1]. For most multidimensional nonseparable systems, periodic and quasiperiodic motion can only exist in a part of the phase space. On the other hand, in one dimension, real Hamiltonian systems are relatively simple and because of the existence of the constant of motion, namely, the total energy, they are integrable and hence closed periodic classical trajectories can always exist.

Recently there has been an increased interest in classical mechanics of complex nonhermitian Hamiltonian systems [2–12]. For these systems classical trajectories usually traverse complex phase space. As in the real phase space, the classical trajectories of one-dimensional (1D) real hermitian systems are also mostly periodic in the complex classical phase space as well. Furthermore, numerical and analytical studies have shown that when energies are real, the classical trajectories of complex Poschl–Teller (PT) symmetric nonhermitian systems are also closed and periodic. A Hamiltonian is PT symmetric if it is invariant under space–time reflection: for P, $p \to -p$, and $x \to -x$; and for T, $p \to -p$, $x \to x$, and $i \to -i$. However, when energies of these systems become complex, the periodic trajectories usually become nonperiodic and open [4, 6]. Recently it was shown that even though almost all the trajectories corresponding to complex energies are open and nonperiodic, for some systems, there are special discrete sets of curves in the complex energy plane for which the trajectories are periodic [13]. On the other hand, in nonhermitian and non-PT symmetric Hamiltonian systems, even for real energies, almost all trajectories except few are nonperiodic and open.

In this paper we investigate the effect of complex time on trajectories of both hermitian and nonhermitian systems. In the next section we show that by introducing time as a complex quantity with specific features, the nonperiodic and open trajectories of complex nonhermitian systems as well as most real hermitian systems with complex energies can become periodic and closed. The phase angles corresponding to several nontrivial complex nonhermitian systems will be determined as illustrations in Sect. 3. Further, in Sect. 4, we show that by introducing time as a complex quantity with specific phase angles, real hermitian systems, such as $H = p^2/2m + x^4 + bx^3 + cx^2 + dx$ (b, c, and d are real quantities and mass of the particle, m, is taken as 1/2) produce periodic classical trajectories with infinitely many discrete real periods for most values of b, c, and d. Concluding remarks are made in Sect. 5.

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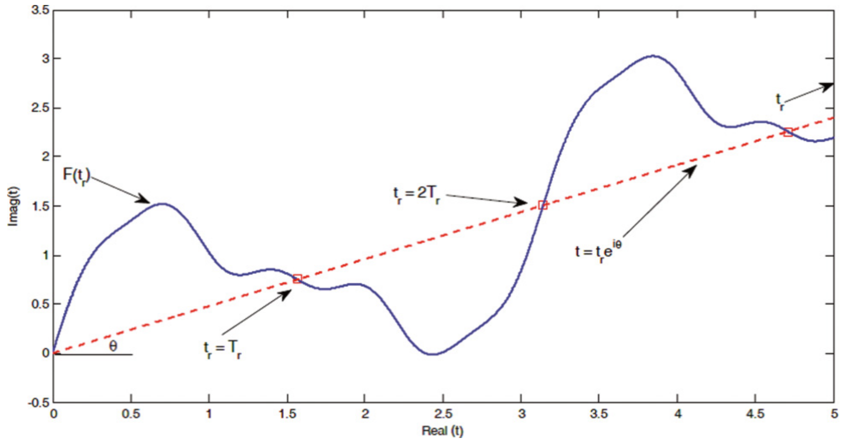
Fig. 1. When time is defined as a complex quantity with a complex function $t = F(t_r)$ (solid line) satisfying the condition $F(nT_r) = nT_r e^{i\tau}$ nonperiodic trajectories become periodic with period T_r. As a special case, complex time can be defined with a fixed phase angle τ such that $t = F(t_r) \equiv t_r e^{i\tau}$ (broken line).



2. Complex time and nonperiodic trajectories

In this section we show that any nonperiodic trajectory with complex period can be made periodic by introducing time as a complex quantity with a specific phase angle. Consider a trajectory $x(t)$ of a one-dimensional (1D) Hamiltonian, H, having a complex period, T. Let $T = T_r e^{i\tau}$ where T_r and τ are the norm and the argument of T, respectively. Then $x(t)$ satisfies the relation

$$x(t_0 + nT) = x\left(t_0 + nT_r e^{i\tau}\right) = x(t_0) \tag{1}$$

where $n\in\mathbb{N}$ and t_0 can be real or complex. When time, t, is a real quantity, as t increases any trajectory starting from $x(0)$ at $t = 0$ will not close itself for any $t > 0$ because real t can never become complex T. Hence, the trajectory is not periodic. However, if we introduce time, t, as a complex quantity with fixed phase angle θ, $(t = t_r e^{i\tau})$ and $x(t)$ is analytically continued for complex t, then as t_r increases it can satisfy $t_r = nT_r$ and

$$x\left(t_r e^{i\tau}\right) = x\left(nT_r e^{i\tau}\right) = x(0) \tag{2}$$

Hence the trajectory is periodic with period T_r. Actually, for any complex time t defined by

$$t = F(t_r) \tag{3}$$

where t_r is real and $F: \mathbb{R} \to \mathbb{C}$ is a continuous complex function with the property that $F(nT_r) = nT_r e^{i\tau}$ for $\forall n \in \mathbb{N}$ as in Fig. 1, the trajectory becomes periodic with period T_r. However, for simplicity, in the rest of this paper, when we introduce complex t, unless otherwise mentioned, t is always taken as $t = F(t_r) \equiv t_r e^{i\tau}\ \forall t_r \in \mathbb{R}$.

The preceding result is valid for both nonperiodic trajectories of hermitian and nonhermitian Hamiltonians. In the next section, with four different Hamiltonians, the complex phase angles and behavior of classical trajectories will be examined.

3. Illustrations

In this section we study four Hamiltonians that possess classical trajectories with complex periods.

First, the complex periods are obtained with both analytical and numerical methods. Then the complex time introduced in the previous section will be utilized to obtain phase angles that make nonperiodic trajectories periodic.

3.1. Complex harmonic oscillator

The first example is the complex harmonic oscillator;

$$H_1(\mu) = p^2 + \frac{1}{2}\mu^2 x^2 \tag{4}$$

For the Hamiltonian, H_1, the classical equation of motion can be solved exactly [14] and $x(t)$ is given by

$$x(t) = \frac{\sqrt{2E}}{\mu}\sin\left(\mu t + \phi_r + i\phi_i\right) \tag{5}$$

where ϕ_r and ϕ_i are real constants that depend on the initial conditions and constant μ can be real or complex. We write $\mu = \mu_0 e^{i\theta_\mu}$ where μ_0 and θ_μ are real numbers with $0 \le \theta_\mu \le 2\pi$. If E is real and positive and $\theta_\mu = 0$ then the x(t) is periodic with a real period $2\pi/\mu$. When μ is complex, $\theta_\mu \neq 0$ and the x(t) is nonperiodic. However, if time, t, is taken as complex with $t = t_r e^{i\tau}$ where $\tau = -\theta_\mu$ then the x(t) becomes periodic with the real period $2\pi/\mu_0$. Note that even when E is complex the trajectories are periodic.

3.2. Anharmonic oscillator $p^2 + \lambda x^N$

When N is even and E and λ are real, the Hamiltonian

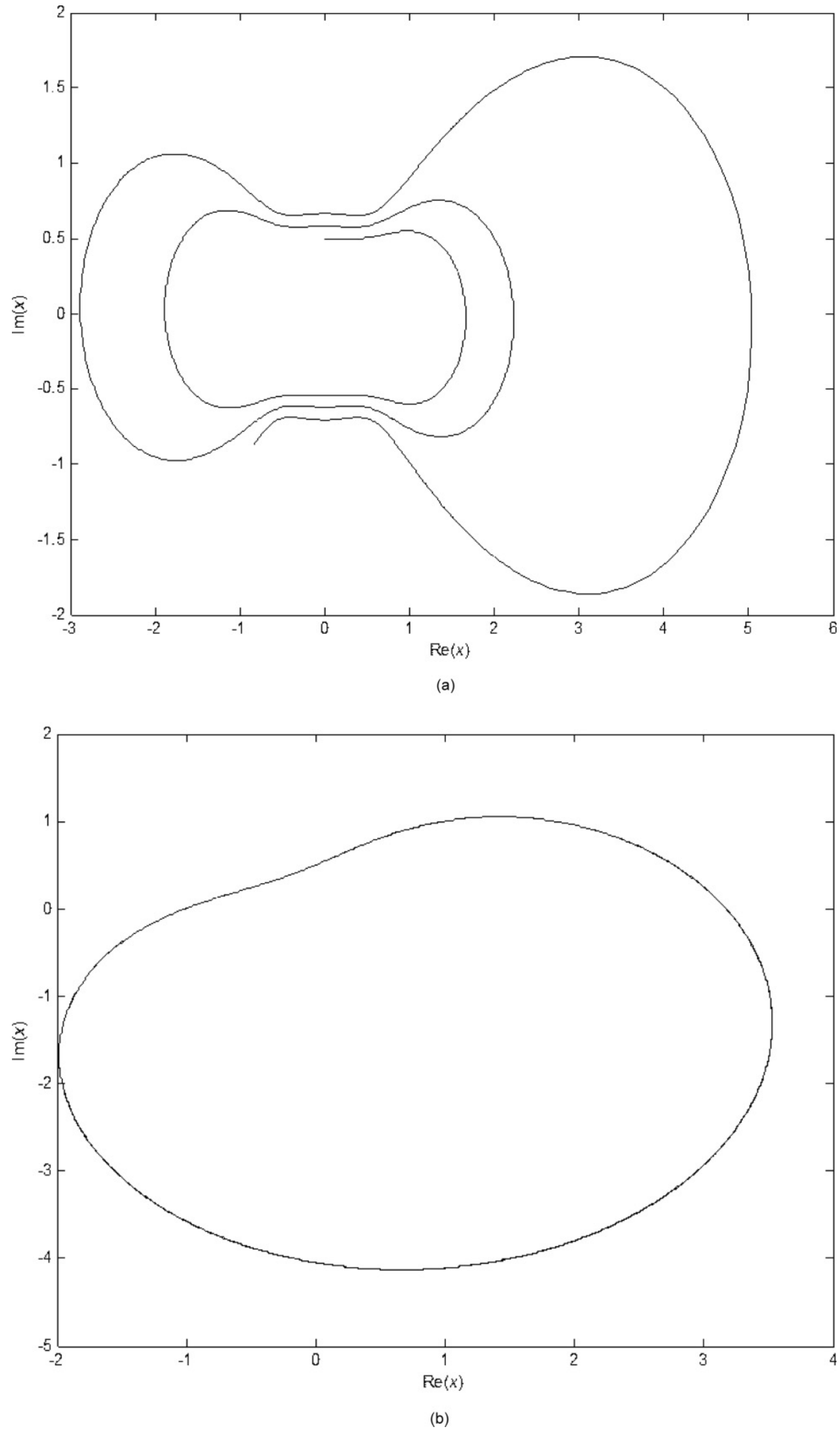
$$H_2(N, \lambda) = p^2 + \lambda x^N \tag{6}$$

has bounded motion and close periodic classical trajectories. However, when λ is complex and (or) E is complex, these trajectories become nonperiodic (Fig. 2*a*). The complex phase angle, τ, for this system can be easily found by rescaling the equation of motion. Consider $H_2(2M, \lambda)$ when λ and E are complex. The equation of motion is

$$\frac{dx}{dt} = \sqrt{E - \lambda x^{2M}} \tag{7}$$

Now x is scaled by a factor $e^{i\theta_x}$ and time t by a factor $e^{i\tau}$ such that (7) involves only real variables and hence the system will have close periodic trajectories. The θ_x is found as

Fig. 2. (*a*) A typical trajectory of the Hamiltonian $H_2(6, \mu) = (p^2/2) + \mu x^6$ where μ is complex, drawn for real time. The trajectory is not periodic. (*b*) The time is taken as a complex quantity with phase angle $\tau = -0.016\,611$. The trajectory has become periodic. The trajectory is drawn for $\mu = 1 + 0.1i$, $x(0) = 0.5i$, and $E = 1.0$.



$$\theta_x = \frac{\arg(E) - \arg(\lambda)}{2M} \tag{8}$$

and complex phase angle τ is determined to be

$$\tau = -\frac{\arg(\lambda) + (M - 1)\arg(E)}{2M} \tag{9}$$

Figure 2*b* shows the same trajectory in Fig. 2*a* when complex time is introduced.

When N is odd, $H_2(N, \lambda)$ does not possess bound states unless λ is complex. Consider the case when $N = 3$. When λ is purely imaginary (i.e., $H_2(3, ia) = (p^2/2) + iax^3$ with real a), this Hamiltonian is PT symmetric and possesses closed periodic classical trajectories. However, when λ or energy E is complex, the trajectories become nonperiodic, and τ in this case is found as [4]

$$\tau = \frac{\pi - 2\arg(\lambda) - \arg(E)}{6} \tag{10}$$

For the Hamiltonians H_1 and H_2, phase angle τ is found in a simple manner. However, finding τ for the next two examples is not a trivial problem.

3.3. Anharmonic oscillator $(1/2)\omega^2x^2 + \mu x^4$

We find the phase angle, τ, for

$$H_3(\omega, \mu) = p^2 + \frac{1}{2}\omega^2x^2 + \mu x^4 \tag{11}$$

using the Hamilton–Jacobi method developed by Capman et al. [15] and Born [16].

$$H_3(\omega, \mu) = H_0(\omega) + \mu x^4 \tag{12}$$

where

$$H_0(\omega) = p^2 + \frac{1}{2}\omega^2x^2 \tag{13}$$

which is exactly solvable. This semiclassical method has been used to investigate 1D nonhermitian systems recently [17]. The Hamilton–Jacobi method is based on the existence of the action variable. We obtain the following equations for the preceding Hamiltonian [17]:

$$E(J) = J\omega + A_0 \tag{14}$$

and

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-ik\theta}\left(\frac{2n+1}{\omega}\right)^2\cos^4\theta \tag{15}$$

where

$$n = \left(J - \frac{1}{2}\right) - \frac{1}{\omega}\sum_{k'} e^{ik'\theta}A_{k'} \tag{16}$$

and the period is given by $T = \partial J/\partial E$. The period is calculated using (14), (15), and (16) and it is approximately obtained as

$$T \simeq \frac{1}{\omega + \bar{A}_0} \tag{17}$$

where

$$\bar{A}_k = \frac{1}{2\pi\omega^2}\int_0^{2\pi} \bar{n}(2n+1)\cos^4\varphi\, e^{-ik\varphi}d\varphi \tag{18}$$

$$\bar{n} = 1 - \frac{1}{\omega}\sum_{k'} e^{ik'\varphi}\bar{A}_{k'} \tag{19}$$

To obtain T from (17), (18) and (19) should be solved iteratively and then A_0 should be calculated. We now define time t to be complex, $t = t_r e^{i\tau}$ with $\tau = \arg(T)$. Then the classical equation of motion is solved numerically with complex t. Fig. 3a shows a typical trajectory that is nonperiodic when μ is complex. This trajectory becomes periodic for the correct phase angle, τ, as shown in Fig. 3b.

3.4. PT potential hole

The Hamiltonian for the PT potential hole is

$$H_4(a, b, \alpha) = p^2 + \frac{a}{\sin^2 \alpha x} + \frac{b}{\cos^2 \alpha x} \tag{20}$$

where $\alpha \in C$ and $a,b \in \mathbb{R}$. When α is real, the Hamiltonian system $H_4(a, b, \alpha)$ is exactly solvable and possesses classical periodic trajectories. Classical action variable J for this system is given by [18]

$$J = \frac{\sqrt{E} - \sqrt{a} - \sqrt{b}}{2\alpha} \tag{21}$$

When α is purely imaginary, this system is PT symmetric and all the quantum energy eigenvalues are real. When α is complex, trajectories are nonperiodic and energy eigenvalues are complex. A typical nonperiodic trajectory of this system for complex α and complex E is shown in Fig. 4a. The period for the preceding system is

$$T = \frac{\partial J}{\partial E} = \frac{1}{4\alpha\sqrt{E}} \tag{22}$$

If α and E are complex, then

$$T = T_r e^{i\tau} = \frac{\exp\{i[\arg(\alpha) + (\arg(E)/2)]\}}{4\alpha_r\sqrt{E_r}} \tag{23}$$

Now $T_r = 1/4\alpha_r\sqrt{E_r}$ and $\tau = -[\arg(x) + (\arg(E)/2)]$. Using $t = t_r e^{i\tau}$ as time, the nonperiodic trajectories of this system can be made periodic as shown in Fig. 4b. In the aforementioned manner we can make nonperiodic trajectories of complex Hamiltonians periodic.

The classical action angle variable w for periodic motion is a linear function of time [14],

$$w = \nu t + \beta = \frac{t}{T} + w_0 \tag{24}$$

where ν is the frequency of the motion, T is the period, and w_0 is the action angle variable at $t = 0$.

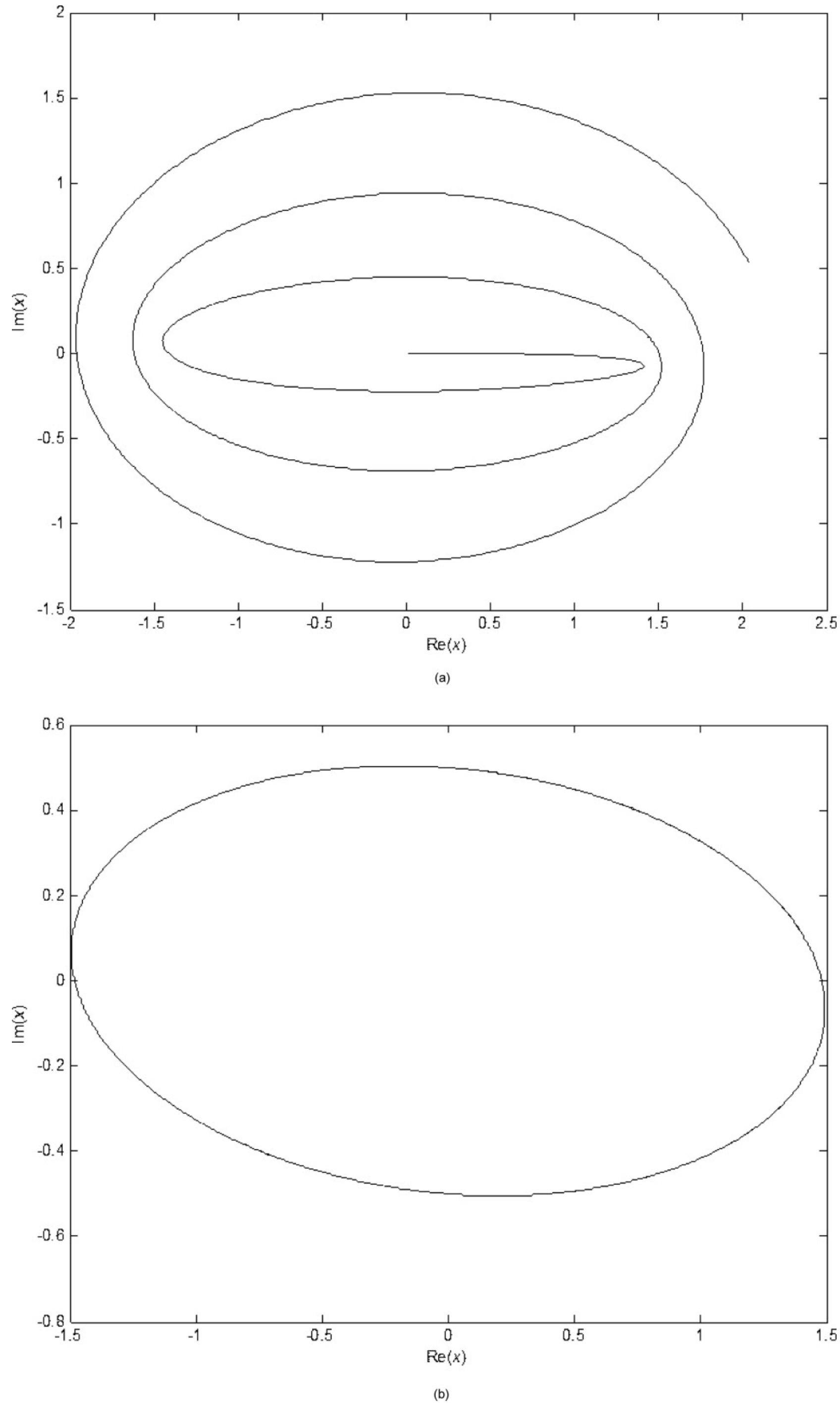
If the motion is nonperiodic with a complex period, w associated with the nonperiodic trajectory is complex and can be obtained by analytically continuing (24) for complex $T = T_r e^{i\tau}$ or complex frequency, ν. It is evident from (24) that by introducing a correct complex phase angle, τ, for time t ($t = t_r e^{i\tau}$), we can make the classical action angle variable, w, a real quantity (assuming w_0 is real) and hence make the motion periodic with real frequency $\nu = 1/T_r$.

4. Complex time and periodic trajectories of hermitian systems

In this section we examine the periodic trajectories of the hermitian Hamiltonian

$$H = \frac{p^2}{2m} + x^4 + bx^3 + cx^2 + dx \tag{25}$$

Fig. 3. (*a*) A typical trajectory of the Hamiltonian $H_3(\omega, \mu) = (p^2/2) + (1/2)\omega^2x^2 + \mu x^4$ where ω is real and μ is complex. The trajectory is not periodic. (*b*) A classical trajectory wherein the time is taken with the complex phase angle $\tau = -0.973\ 014\ 7$. The trajectory is now periodic. The trajectory is drawn for $\omega = 1.0$ and $\mu = 0.01i$, $x(0) = 0.5i$, and $E = 2.5 + 0.1i$.



where b, c, and d are real parameters and $m = 1/2$. The equation of motion for the preceding potential is

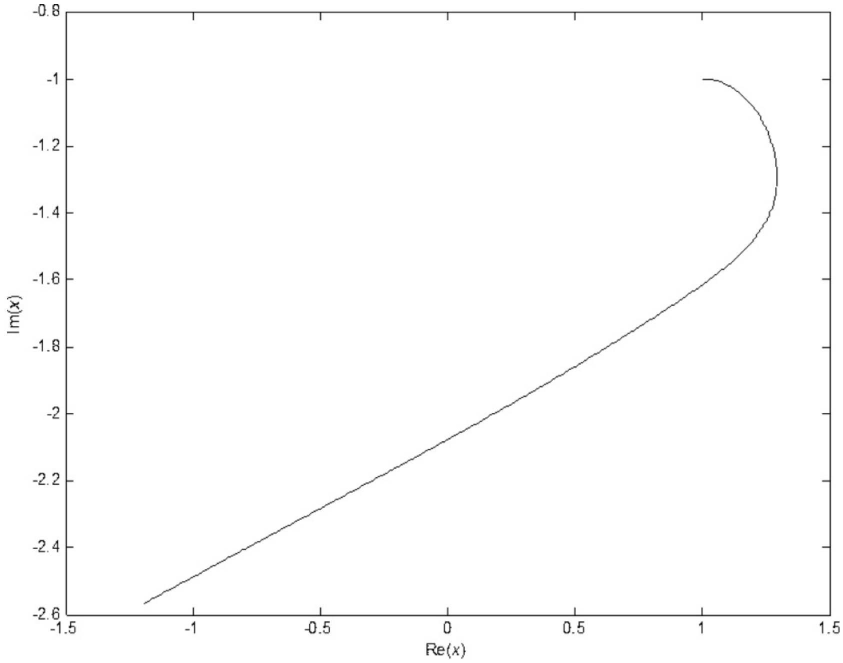
$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = 2\sqrt{E - x^4 - bx^3 - cx^2 - dx} \tag{26}$$

The turning points of this system are taken as x_0, x_1, x_2, and x_3. Integrating (26) we have

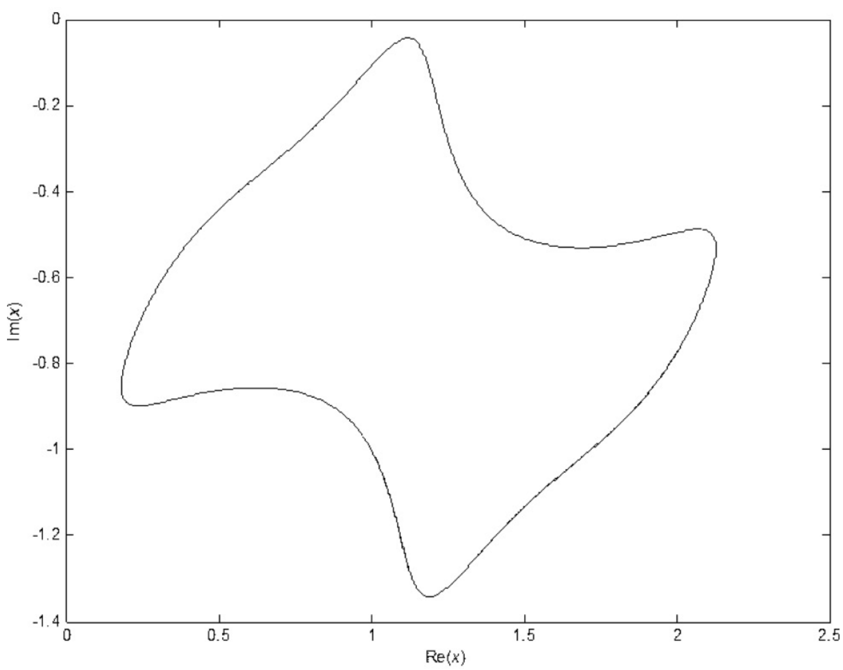
$$\int \frac{\mathrm{d}x}{\sqrt{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}} = 2\mathrm{e}^{i\pi/2}t + c \tag{27}$$

where c is the constant of integration, which depends on initial conditions. The left-hand side of the preceding equation is an elliptic integral of the first kind and hence (27) becomes

Fig. 4. The trajectory is drawn for the PT Hamiltonian with $a = 2.0$, $b = 1.0$, and $\alpha = 1.0 + 0.6i$ for complex energy $E = 1.0 + i$ and $x_0 = 1 - i$. (*a*) The Hamiltonian is complex nonhermitian and the trajectory is nonperiodic and escaping to ∞. (*b*) Time is taken as a complex quantity with the argument $\tau = -0.933\,119$. Now the trajectory has become periodic.



(a)



(b)

$$\frac{2}{\sqrt{(x_0 - x_2)(x_1 - x_3)}} \times F\left(\sin^{-1}\left[\sqrt{\frac{(x - x_1)(x_0 - x_2)}{(x - x_0)(x_1 - x_2)}}\right], \frac{(x_1 - x_2)(x_0 - x_3)}{(x_0 - x_2)(x_1 - x_3)}\right) = 2e^{i\pi/2}t + c \tag{28}$$

where F is an elliptic function [19, 20]. We invert the preceding equation in terms of Jacobian elliptic function sn as

$$x(t) = \frac{x_1(x_0 - x_2) - x_0(x_1 - x_2)\mathrm{sn}^2(u)}{(x_0 - x_2) - (x_1 - x_2)\mathrm{sn}^2(u)} \tag{29}$$

where

$$u = \sqrt{(x_0 - x_2)(x_1 - x_3)}e^{i\pi/2}t + \alpha'$$

[19] and modulus

$$k = \left(\frac{(x_1 - x_2)(x_0 - x_3)}{(x_0 - x_2)(x_1 - x_3)}\right)^{1/2}$$

and α' is an arbitrary constant that is determined by the initial conditions. Also note that $x(t)$ in the preceding equation is still a solution of (26), when x_0, x_1, x_2, and x_3 are interchanged in any order (e.g., $x_3 \to x_0 \to x_1 \to x_2 \to x_3$). To understand how the trajectories behave, we need to recognize the periodic, bounded and unbounded properties of the function $x(t)$. First we find complementary modulus k' and complete elliptic functions K and K'. They are defined by

$$k'^2 = 1 - k^2 = \frac{(x_0 - x_1)(x_2 - x_3)}{(x_0 - x_2)(x_1 - x_3)} \tag{30}$$

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2(\phi))^{-1/2} \mathrm{d}\phi \tag{31}$$

$$K' = \int_0^{1} (1 - t^2)^{-1/2} (1 - k'^2 t^2)^{-1/2} \mathrm{d}t \tag{32}$$

K and K' are evaluated directly from the preceding equations. Because α in (29) determines the trajectory, trajectories become bounded or unbounded depending on the value of α. Jacobian elliptic function sn is a doubly periodic function of u with periods $4K$ and $2iK'$. It is analytic except at the points congruent to iK' or to $2K + iK'$ (mod $4K$, $2iK'$). These points are simple poles. We can relate the periodicity of Jacobian elliptic function $\mathrm{sn}(u)$ to the periodic motion of the trajectories. The condition for a trajectory to become periodic is

$$\sqrt{(x_0 - x_2)(x_1 - x_3)}\,\mathrm{e}^{i\pi/2} T = 4nK + 2miK' \tag{33}$$

for some integers n and m. Then the period is given by

$$T = n\alpha + m\beta \tag{34}$$

where

$$\alpha = \frac{4K\mathrm{e}^{-i\pi/2}}{\sqrt{(x_0 - x_2)(x_1 - x_3)}} \tag{35}$$

$$\beta = \frac{2iK'\mathrm{e}^{-i\pi/2}}{\sqrt{(x_0 - x_2)(x_1 - x_3)}} \tag{36}$$

If T is real for some integers n and m then the motion is periodic. T depends on the parameters b, c, d, and energy, E, through the turning points x_0, x_1, x_2, and x_3. If T is complex, trajectories are not periodic. Let $T = T_r \mathrm{e}^{i\tau}$. Now time t is taken as a complex quantity as $t = t_r \mathrm{e}^{i\tau}$. As we have shown in Sect. 2, trajectories are now periodic with periods T_r where

$$T_r(n, m) = \sqrt{[n\mathrm{Re}(\alpha) + m\mathrm{Re}(\beta)]^2 + [n\mathrm{Im}(\alpha) + m\mathrm{Im}(\beta)]^2} \tag{37}$$

and

$$\tau(n, m) = \tan^{-1}\left(\frac{n\mathrm{Im}(\alpha) + m\mathrm{Im}(\beta)}{n\mathrm{Re}(\alpha) + m\mathrm{Re}(\beta)}\right) \tag{38}$$

Because n and m are integers, only discrete values of τ will produce periodic trajectories. For real time, the Hamiltonian in (25) possesses only one real period. However, for real parameter values (b, c, d, and energy E), if time is complex, there are infinitely many discrete values of τ for which the classical trajectories of this Hamiltonian are periodic and there are infinitely many distinct real periods corresponding to these τ values.

5. Concluding remarks

In this paper we showed that when time is taken as a complex quantity with a fixed phase angle or as a complex function having specific properties, nonperiodic open trajectories will become periodic and closed. The nonperiodic nature of the original trajectory may be due to complex parameters of the potential or complex energy. With four examples we demonstrated the preceding claim. When energy and parameters b, c, and d of the Hamiltonian $H = p^2/2 + x^4 + bx^3 + cx^2 + dx$ are real, we showed that there are infinitely many discrete values of phase angle τ for which trajectories are closed and periodic. For these τ values there are infinitely many distinct periods associated with the periodic trajectories. Although detailed discussion on the consequence of complex time on classical trajectories was limited to five illustrations in this paper, due to the general manner, complex time is defined in (3) and can be applied to any 1D classical trajectory having a complex period.

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