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Isospectral Hermitian counterpart of complex non Hermitian Hamiltonian $p^2 - gx^4 + a/x^2$

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In this paper we show that the non-Hermitian Hamiltonians $H = p^2 - gx^4 + a/x^2$ and the conventional Hermitian Hamiltonians $h = p^2 + 4gx^4 + bx$ $(a, b \in \mathbb{R})$ are isospectral if $a = (b^2 - 4g\hbar^2)/16g$ and $a \geq -\hbar^2/4$. This new class includes the equivalent non-Hermitian -Hermitian Hamiltonian pair, $p^2 - gx^4$ and $p^2 + 4gx^4 - 2\hbar\sqrt{g}x$, found by Jones and Mateo six years ago as a special case. When $a = \left(b^2 - 4g\hbar^2\right)/16g$ and $a < -\hbar^2/4$, although h and H are still isospectral, b is complex and h is no longer the Hermitian counterpart of H.

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I. INTRODUCTION

Bender and Boettcher in a pioneering paper [1] showed that non Hermitian, PT-symmetric Hamiltonians of the form

$$H_0 = p^2 - g\,(ix)^N \tag{1}$$

posses real and positive eigenspectra when $N \geq 2$. Since then many 1-D PT-symmetric non Hermitian Hamiltonian models have been investigated both quantum mechanically as well as classically. Interest in non-Hermitian PT-symmetric models has increased considerably during the last decade mainly due to their usefulness in the areas such as particle-physics, quantum optics, supersymmetric and magnetohydrodynamics and now the applicability and the usefulness of non-Hermitian PT-symmetric quantum mechanics have been well established [1–8]. If PT-symmetry is not spontaneously broken, Non-Hermitian PT-symmetric Hamiltonians have real energy spectra. However, for a given PT-symmetric Hamiltonian, there is no simple way of figuring out ahead of time whether the PT-symmetry is spontaneously broken or not. Mostafazadeh [6] has proved that if the Hamiltonian of a quantum system possesses an exact PT-symmetry (unbroken PT-symmetry) then the Hamiltonian is equivalent to a Hermitian Hamiltonian which has the same spectrum. This was achieved by constructing the unitary operator relating a given non-Hermitian Hamiltonian with exact PT-symmetry to a Hermitian Hamiltonian. Nonetheless, only in a few instances, people succeeded in finding Hermitian Hamiltonians which posses the same eigenspectra as PT-symmetric non Hermitian Hamiltonians [7–12].

Using operator techniques and path integral methods, Jones et al [7–9] found that the complex non-Hermitian PT-symmetric Hamiltonian $p^2 - gx^4$ and the conventional Hermitian Hamiltonian $p^2 + 4gx^4 - 2\sqrt{g}x$ are isospectral. However using a method based on a combination of certain integrals (viz. Fourier) and point (i.e. change-of-variables) spectrum-preserving transformations, Buslaev and Grecchi [10] had shown this equivalence relation several years earlier. It is also

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interesting to note that these results had been published five years earlier than the pioneering paper [1] on PT-symmetry by Bender et al.

Recently, the Asymptotic Energy Expansion (AEE) method has been applied by Nanayakkara et al to show that the complex non-Hermitian PT-symmetric Hamiltonian $p^2 - gx^4 + 4i\hbar\sqrt{g}x$ and the conventional Hermitian Hamiltonian $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ have the same eigenspectra [12].

In this paper we show that the Hamiltonians $H = p^2 - gx^4 + a/x^2$ and $h = p^2 + 4gx^4 + bx$ are isospectral if $a = \left(b^2 - 4g\hbar^2\right)/16g$ and the $p^2 - gx^4 + 4i\hbar\sqrt{g}x$ and $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ as well as $p^2 - gx^4$ and $p^2 + 4gx^4 - 2\hbar\sqrt{g}x$ are special cases of H and h. The outline of the paper is as follows. In Sec. II, the equivalence condition for H and h is derived using the AEE method. The behavior of eigenenergies and breakdown of PT-symmetry with respect to the parameters of the Hamiltonians are investigated in Sec. III. Exact ground state wave functions, superpotentials and supersymmetric partners of both Hamiltonians are analyzed in Sec. IV. Concluding remarks are given in Sec. V.

II. DERIVATION OF EQUIVALENCE CONDITION

In this section, we establish the conditions for which the non-Hermitian PT- symmetric quartic Hamiltonian

$$H = p^2 - gx^4 + \frac{a}{x^2} \tag{2}$$

and the conventional Hermitian Hamiltonian

$$h = p^2 + \alpha x^4 + bx \tag{3}$$

are equivalent. Here a, g, α and b are assumed to be real. However, later in the Sec. III, we consider the cases where these parameters are complex as well. In a previous study on equivalent non Hermitian and Hermitian Hamiltonians [12], it was shown that the Hamiltonians $p^2 - gx^4 + 4i\hbar\sqrt{g}x$ and $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ are equivalent with zero energy ground states and the supersymmetric partner of $-gx^4 + 4i\hbar\sqrt{g}x$ is $-gx^4 + \frac{2\hbar^2}{x^2}$. Further these two Hamiltonians are found to be isospectral as well. Consequently, the Hamiltonian $p^2 - gx^4 + \frac{2\hbar^2}{x^2}$ is equivalent to $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ and, hence $a = 2\hbar^2, \alpha = 4g$, and $b = 6\hbar\sqrt{g}$ are one set of parameters for which (2) and (3) are equivalent. Therefore, it is worthwhile to investigate whether there are any other parameter values for which H and h are equivalent.

In order to obtain the general conditions of equivalence, we used the Asymptotic Energy Expansion (AEE) method [12–14] which is employed by Nanayakkara et al. The AEE method is an analytic method where each term in the expansion can be obtained explicitly in terms of Gamma functions and multinomials of the parameters in the potential. The accuracy and the applicability of AEE method to obtain equivalent Hamiltonians have been demonstrated in [12]. First the AEE is derived for the non Hermitian Hamiltonian H. Since Hamiltonian H contains a $1/x^2$ term, the standard AEE method used for even degree polynomial potentials has to be modified. Therefore the complete derivation is described below.

Consider the non Hermitian Hamiltonian H

$$H\left(x,p\right)=p^{2}+V\left(x\right) \tag{4}$$

where $V\left(x\right)=-gx^{4}+\frac{a}{x^{2}}$.

The AEE quantization condition for this potential is

$$J\left(E\right)=n\hbar \tag{5}$$

where n is a positive integer and quantum action variable $J\left(E\right)$ is given by

$$J\left(E\right)=\frac{1}{2\pi}\int_{\gamma}P\left(x,E\right)dx \tag{6}$$

$P\left(x,E\right)$ satisfies the Riccati equation

$$\frac{\hbar}{i}\frac{\partial P\left(x,E\right)}{\partial x}+P^{2}\left(x,E\right)=E-V\left(x\right)=P_{c}\left(x,E\right) \tag{7}$$

Note that $P\left(x,E\right)$ relates to the wave function as $P\left(x,E\right)=\frac{\hbar}{i}\frac{\partial\Psi/\partial x}{\Psi}$. The contour γ in (6) encloses two physical turning points of $P_{c}\left(x,E\right)$. Boundary conditions imposed upon $P\left(x,E\right)$ is $P\left(x,E\right)\rightarrow P_{c}\left(x,E\right)$ as $\hbar\rightarrow 0$ [15, 16].

For the above potential, (7) becomes

$$\frac{\hbar}{i}\frac{\partial P\left(x,E\right)}{\partial x}+p^{2}\left(x,E\right)=E+gx^{4}-\frac{a}{x^{2}}. \tag{8}$$

Let $\epsilon=E^{-1/4}$ and $y=g^{1/4}\epsilon x$. Then (8) becomes, after simplification,

$$\hat{h}y^{2}\epsilon^{5}\frac{\partial P\left(y,\epsilon\right)}{\partial y}+y^{2}\epsilon^{4}P^{2}\left(y,\epsilon\right)=y^{2}(1+y^{4})-ag^{1/2}\epsilon^{6} \tag{9}$$

where $\hat{h}=\frac{\hbar}{i}g^{1/4}$. In order to obtain asymptotic energy expansion, first $P\left(y,\epsilon\right)$ is expanded as an asymptotic series in powers of ϵ and subsequently obtain recurrence relations. This expansion usually has zero radius of convergence. However, truncating the series after a finite number of terms provides a good approximation to $P\left(y,\epsilon\right)$ [17, 18]. The asymptotic series expansion is written as

$$P\left(y,\epsilon\right)=\epsilon^{s}\sum_{k=0}^{\infty}a_{k}\left(y\right)\epsilon^{k} \tag{10}$$

where a_k and s are determined below. Substituting (10) in (9) and equating coefficients of ϵ^{0}, s and a_0 are found as $s=-2$ and $a_0=\sqrt{1+y^{4}}$ and (9) becomes

$$\hat{h}y^{2}\sum_{k=0}^{\infty}\epsilon^{k+3}\frac{da_{k}}{dy}+y^{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{i}a_{j}\epsilon^{i+j}=y^{2}(1+y^{4})-ag^{1/2}\epsilon^{6} \tag{11}$$

Next assume $a_k=0$ when $k<0$ and rearranging terms,

$$\left(\hat{h}y^{2}\sum_{k=1}^{\infty}\frac{da_{k-3}}{dy}+y^{2}\sum_{k=1}^{\infty}\sum_{i=1}^{k-1}a_{i}a_{k-i}+2y^{2}a_{0}\sum_{k=0}^{\infty}a_{k}\right)\epsilon^{k}=y^{2}(1+y^{4})-ag^{1/2}\epsilon^{6}. \tag{12}$$

Then coefficients a_k's are given by

$$a_k = \frac{-1}{2y^2 a_0}\left[y^2 \sum_{i=1}^{k-1} a_i a_{k-i} + \hat{h} y^2 \frac{da_{k-3}}{dy} + a g^{1/2} \delta_{k,6}\right]. \tag{13}$$

In the above formula $a_k = 0 \ \forall k < 0$. Now J can be written as

$$J\left(E\right) = \sum_{k=0}^{\infty} b_k E^{\frac{-(k-3)}{4}} \tag{14}$$

where

$$b_k = \frac{1}{2\pi} \int\limits_{\gamma} a_k dy \tag{15}$$

and can be determined analytically in terms of g and a. The contour γ encloses the two branch points of $\sqrt{1+y^4}$ (i.e. $e^{i\pi/4}$ and $e^{3i\pi/4}$) on the complex plane. The quantization condition $J\left(E\right) = n\hbar$ determines the eigenenergies of H.

Using (13) and evaluating the integral (15) analytically, the asymptotic series is obtained. The eigenenergy expansion becomes

$$J\left(E\right) = \sum_{k=0}^{\infty} b_k E^{\frac{-(k-3)}{4}}. \tag{16}$$

Here first six non zero b_k's are

$$b_0 = \frac{\Gamma\left[\frac{1}{4}\right]}{3 g^{1/4} \sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]}, \tag{17}$$

$$b_3 = -\frac{\hbar}{2}, \tag{18}$$

$$b_6 = \frac{g^{1/4}(4a - \hbar^2)\ \Gamma\left[\frac{3}{4}\right]}{4\sqrt{2\pi}\ \Gamma\left[\frac{1}{4}\right]}, \tag{19}$$

$$b_{12} = \frac{g^{3/4}(80a^2 - 200ah^2 - 11\hbar^4)\ \Gamma\left[\frac{1}{4}\right]}{1536\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]}, \tag{20}$$

$$b_{18} = -\frac{77 g^{5/4}(192a^3 - 1296a^2h^2 + 1860ah^4 + 61\hbar^6)\ \Gamma\left[\frac{3}{4}\right]}{30720\sqrt{2\pi}\ \Gamma\left[\frac{1}{4}\right]}, \tag{21}$$

$$b_{24} = -\frac{1105 g^{7/4}(256a^4 - 3328a^3h^2 + 14432a^2h^4 - 17360a\hbar^6 + 353h^8)\ \Gamma\left[\frac{1}{4}\right]}{3670016\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]}. \tag{22}$$

The next step is to obtain the AEE expansion for the Hamiltonian h in (3). Since the AEE expansion for h has been derived in [12], only the result is presented below. The expansion of the quantum action variable $J(E)$ for the Hamiltonian h is

$$J'\left(E\right)=\sum_{k=0}^{\infty}\beta_k E^{\frac{-(k-3)}{4}}. \tag{23}$$

The first six non zero β_k's are

$$\beta_0=\frac{\Gamma\left[\frac{1}{4}\right]}{3\sqrt{\pi}\alpha^{1/4}\,\Gamma\left[\frac{3}{4}\right]}, \tag{24}$$

$$\beta_3=-\frac{\hbar}{2}, \tag{25}$$

$$\beta_6=-\frac{\left(2\hbar^2\alpha-b^2\right)\,\Gamma\left[\frac{3}{4}\right]}{8\sqrt{\pi}\alpha^{3/4}\,\Gamma\left[\frac{1}{4}\right]}, \tag{26}$$

$$\beta_{12}=\frac{\left(44\hbar^4\alpha^2-60\hbar^2\alpha b^2+5b^4\right)\,\Gamma\left[\frac{1}{4}\right]}{6144\sqrt{\pi}\alpha^{5/4}\,\Gamma\left[\frac{3}{4}\right]}, \tag{27}$$

$$\beta_{18}=\frac{77\left(488\hbar^6\alpha^3-636\hbar^4\alpha^2b^2+90\hbar^2\alpha b^4-3b^6\right)\,\Gamma\left[\frac{3}{4}\right]}{245760\sqrt{\pi}\alpha^{7/4}\,\Gamma\left[\frac{1}{4}\right]}, \tag{28}$$

$$\beta_{24}=-\frac{1105\left(5648\hbar^8\alpha^4-6304\hbar^6\alpha^3b^2+1064\hbar^4\alpha^2b^4-56\hbar^2\alpha b^6+b^8\right)\,\Gamma\left[\frac{1}{4}\right]}{58720256\sqrt{\pi}\alpha^{9/4}\,\Gamma\left[\frac{3}{4}\right]}. \tag{29}$$

By equating the coefficients of $J(E)$ expansions of both Hamiltonians, the conditions of the equivalence are obtained as

$$\alpha=4g, \tag{30}$$

$$a=\frac{b^2-4g\hbar^2}{16g}. \tag{31}$$

The condition (30) is obtained by equating terms b_0 and β_0 while condition (31) is derived by equating b_6 and β_6. When these two conditions are satisfied, it was found that b_k and β_k are equal for next hundred k values indicating AEE of $J(E)$ and $J'(E)$ identical. In addition, by imposing the condition that h is Hermitian, the parameters a and b become $b^2 \geq 0$ and $a \geq -\frac{\hbar^2}{4}$.

Since the AEE expansion is accurate for higher eigenvalues, we have verified the equivalence of the Hamiltonians h and H for low energies by solving the Schrödinger equation numerically along suitable contours for various values of parameters a and b.

It is evident from the Table I and Table II that both Hamiltonians h and H have the same eigenspectra for first ten eigenstates. On the other hand, the expansion of $J(E)$ is very accurate for large eigenvalues and both Hamiltonians have the identical $J(E)$ expansions as shown above.

n	E_H	E_h	E_J
0	-2.4558329	-2.4558327	1.5186675
1	4.5014539	4.5014546	4.5046982
2	10.931991	10.931992	10.931992
3	17.793015	17.793016	17.793016
4	25.238132	25.238134	25.238134
5	33.213971	33.213972	33.213972
6	41.666149	41.666150	41.666150
7	50.549802	50.549804	50.549804
8	59.828456	59.828459	59.828459
9	69.472108	69.472110	69.472110
10	79.455684	79.455685	79.455685

TABLE I: Verification of the equivalence of Hamiltonians $H = p^2 - x^4 + \frac{6}{x^2}$ and $h = p^2 + 4x^4 + 10x$. The first ten exact eigenenergy values of H and h and approximate eigenvalues E_J obtained by $J(E)$ expansion method are given up to eight digits.

n	E_H	E_h	E_J
0	1.8961344	1.8961346	2.4545618
1	6.0533268	6.0533273	6.0884046
2	11.867933	11.867933	11.866200
3	18.510801	18.510802	18.510890
4	25.836222	25.836224	25.836220
5	33.733312	33.733314	33.733314
6	42.128813	42.128814	42.128814
7	50.969273	50.969275	50.969275
8	60.213679	60.213680	60.213680
9	69.829366	69.829368	69.829368
10	79.789590	79.789590	79.789590

TABLE II: Verification of the equivalence of Hamiltonians $H = p^2 - x^4 - \frac{1}{2x^2}$ and $h = p^2 + 4x^4 + 2ix$. The first ten exact eigenenergy values of H and h and approximate eigenvalues E_J obtained by $J(E)$ expansion method are given up to eight digits.

III. BEHAVIOR OF EIGENENERGIES AND HERMITICITY

In the previous section, the conditions of equivalence have been established. Next the behavior of the eigenvalues of H is examined as a function of the parameter a. The Hermitian condition on h is relaxed such that b^2 can also be negative. Therefore now a can be less than $-\frac{\hbar^2}{4}$ as well. When a is large ($\simeq 40$ and $\hbar = 1$) lower eigenvalues of H are negative as shown in figure 1. As a decreases eigenvalues become larger and whole spectrum become real and positive when $-2.76 < a < 2$. When $a = 2$, H has a zero energy ground state and Hamiltonians H and h are recognized as the equivalent Hamiltonians found by Nanayakkara et al [12]. When $a = 0$, $b = 2g\hbar$, the Hamiltonians H and h become the equivalent non-Hermitian - Hermitian Hamiltonian pair found by Jones et al [7, 8].

If $-\frac{\hbar^2}{4} \leq a < \infty$, h is the Hermitian equivalent Hamiltonian of the PT-symmetric Hamiltonian H. When $a < -\frac{\hbar^2}{4}$, b is pure imaginary and h loses its Hermiticity and becomes non-Hermitian and PT-symmetric. However, h and H are still isospectral. Hamiltonian h for this case has been studied in detail by Delabaere et al [19] and Bender et al [21] in the past. Similar to what Bender et al have observed for the Hamiltonian h, as a decreases below $-\frac{\hbar^2}{4}$ further, adjacent pairs of energy levels of H also coalesce and then become complex, starting with the ground state and the first excited state as shown in figure 1. The value of a at which this coalescence takes place for H is $a = A = -2.76\hbar^2$. Note that when $a < -\frac{\hbar^2}{4}$, decrease in a in the Hamiltonian H is equivalent to an increase in $|b|$ in h.

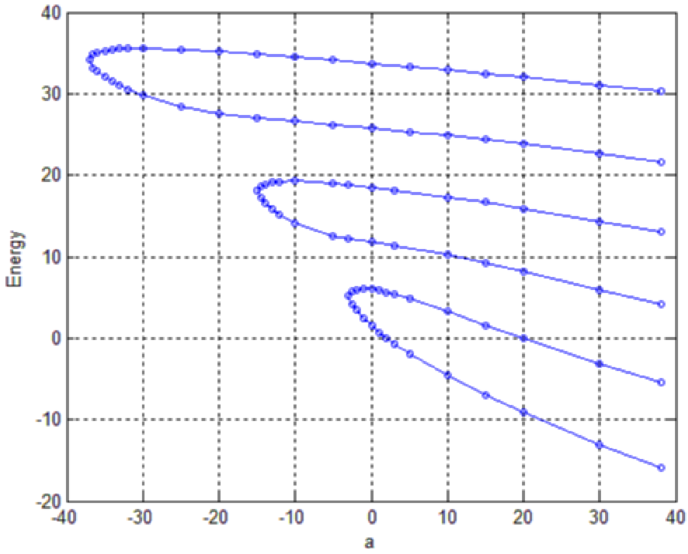

FIG. 1: Six lowest eigenvalues of the Hamiltonian H as a function of the parameter a, when $\hbar = 1$.

At this point it is useful to pay our attention to the Hermiticity of both systems. We have observed previously, for $-\frac{\hbar^2}{4} \leq a < \infty$, h is the Hermitian equivalent of H and both Hamiltonians have real spectra. When $A < a < -\frac{\hbar^2}{4}$, the eigenspectrum of H is real and positive while h has become non-Hermitian and PT-symmetric as b is pure imaginary. Therefore h is no longer the Hermitian equivalent of H. If the PT-symmetry of H is not broken for $A < a < -\frac{\hbar^2}{4}$ then by reference [6], there exists an equivalent Hermitian Hamiltonian which is different from h. However, there is another possibility that although the eigen spectrum of H is entirely real, PT- symmetry of H may be spontaneously broken and therefore H is no longer having a Hermitian counterpart (Note that it has not been proven that real eigenspectra of a PT-symmetric system implies unbroken PT-symmetry) . On the other hand when $a < A$, the lower eigenenergies of both H and h become complex and hence H is no longer has the true PT-symmetry.

IV. UNBROKEN SUPERSYMMETRY

In [12], it was shown that the Hamiltonian

$$H_1 = p^2 - x^4 + 4ix \tag{32}$$

has zero energy ground state and the Hamiltonian

$$H_2 = p^2 - x^4 + 2/x^2 \tag{33}$$

is the supersymmetric partner potential (assume $\hbar = 1$, $2m = 1$, and $g = 1$). In this section we examine these two systems in detail. In a single framework these two systems have been investigated in detail by Dorey et al [20]. H_1 corresponds to $l=0$ and $\alpha=4$ and H_2 corresponds to $l=1$ and $\alpha=0$ in their notations. Therefore our discussion will be based on some of the results they have obtained in [20]. With the above choice of α and l , the ground state wave function of H_1 is on the line $\alpha_-=0$ while the same of H_2 is on the line $\alpha_+=0$ in their notations. Based on [20] and the current study, the following results can be listed;

(1) Hamiltonians H_1 and H_2 have zero energy ground states with the normalizable wave functions $\Phi_0^{(1)}(x)$ and $\Phi_0^{(2)}(x)$ respectively as

$$\Phi_0^{(1)}(x) = ixe^{\frac{i}{3}x^3} \tag{34}$$

and

$$\Phi_0^{(2)}(x) = (ix)^{-1}\, e^{-\frac{i}{3}x^3} \tag{35}$$

where quantization contour starts and ends at $|x|=\infty$ joining the (stokes) sectors S_{-1} and S_1 and

$$S_k = \left\{ x : \left| \arg(x) - \frac{\pi k}{3} \right| < \frac{\pi}{6} \right\} \tag{36}$$

(2) The superpotential $W_{H_1}(x)$ obtained from the zero energy ground state wave function of H_1 is

$$W_{H_1}(x) = -\frac{1+ix^3}{x} \tag{37}$$

(3) The superpotential $W_{H_2}(x)$ obtained from the zero energy ground state wave function of H_2 is

$$W_{H_2}(x) = \frac{1+ix^3}{x} = -W_{H_1}(x) \tag{38}$$

(4) The supersymmetric partner Hamiltonian of H_1 is H_2 and the supersymmetric partner Hamiltonian of H_2 is H_1 hinting at broken supersymmetry. But both have normalizable ground state wave functions assuring unbroken supersymmetry.

Therefore H_1 and H_2 are isospectral having unbroken supersymmetry with zero energy ground states as concluded in [20]. Similar behavior has also been observed for some other systems by Znojil et al [22].

V. SUMMARY AND CONCLUDING REMARKS

In this paper we have shown that the non-Hermitian Hamiltonian $H = p^2 - gx^4 + a/x^2$ is equivalent to the Hermitian Hamiltonian $h = p^2 + 4gx^4 + bx$ if $a = \left(b^2 - 4g\hbar^2\right)/16g$ and $a \geq -\frac{\hbar^2}{4}$. We applied the asymptotic energy expansion (AEE) method to obtain the above result. The AEE method is based on series expansion of the quantum action variables $J(E)$ in rational powers of reciprocal of energy. The $J(E)$ expansions of these two Hamiltonians were found to be identical. In addition, the spectral equivalence of H and h was verified with eigenspectra obtained by solving the Schrödinger equation for these Hamiltonians numerically along suitable contours of integration for various values of a and b.

When $a < -\frac{\hbar^2}{4}$, it was shown that h becomes non-Hermitian and is no longer the Hermitian equivalent of H. However, H and h remain isospectral partners even if $a < -\frac{\hbar^2}{4}$. When a decreases below $a = -2.76\hbar^2$, adjacent pairs of energy levels of H coalesce and then become complex conjugate pairs, starting with the ground state and the first excited state.

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