# Asymptotic behavior of eigenenergies of nonpolynomial oscillator potentials $V(x) = x^{2N} + (\lambda x^{m_1})/(1 + gx^{m_2})$

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**Abstract:** Analytic semiclassical energy expansions of nonpolynomial oscillator (NPO) potentials $V(x) = x^{2N} + (\lambda x^{m_1})/(1 + gx^{m_2})$ are obtained for arbitrary positive integers $N$, $m_1$, and $m_2$, and the real parameters $\lambda$ and $g$ using the asymptotic energy expansion (AEE) method. Because the AEE method has been previously developed only for polynomial potentials, the method is extended with new types of recurrence relations. It is then applied to the preceding general NPO to obtain expressions for quantum action variable $J$ in terms of $E$ and the parameters of the potential. These expansions are power series in energy and the coefficients of the series contain parameters $\lambda$ and $g$ explicitly. To avoid the singularities in the potential we only consider the cases where both $\lambda$ and $g$ are non-negative at the same time. Using the AEE expressions, it is shown that, for certain classes of NPOs, if potentials have the same $N$, and the same $m_1 - m_2$ or $m_1 - 2m_2$ then they have the same asymptotic eigenspectra. It was also shown that for certain cases, both $\lambda$ and $-\lambda$ as well as $g$ and $-g$ will produce the same asymptotic energy spectra. Analytic expressions are also derived for asymptotic level spacings of general NPOs in terms of $\lambda$ and $g$.

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**Résumé :** Utilisant la méthode de développement asymptotique en énergie (AEE), nous obtenons des développements analytiques pour les énergies semi-classiques de potentiels d'oscillateurs non polynomiaux (NPO) $V(x) = x^{2N} + (\lambda x^{m_1})/(1 + gx^{m_2})$, pour toute valeur positive entière de $N$, $m_1$ et $m_2$ et réelle de $\lambda$ et $g$. Puisque la méthode AEE a été développée pour des potentiels polynomiaux, nous devons la généraliser avec de nouvelles relations de récurrence. Nous l'appliquons alors au NPO généralisé ci-dessus afin d'obtenir l'action quantique $J$ en fonction de $E$ et des paramètres du potentiel. Ces développements sont des séries de puissance en énergie et les coefficients des séries contiennent les paramètres $\lambda$ et $g$ explicitement. Afin d'éviter les singularités du potentiel, nous ne considérons que les cas où $\lambda$ et $g$ ne sont pas simultanément négatifs. Utilisant les expressions AEE, nous montrons que pour certaines classes de NPO, si les potentiels ont le même $N$ et les mêmes $m_1 - m_2$ ou $m_1 - 2m_2$, alors ils ont le même spectre asymptotique. Nous montrons aussi que pour certaines classes, à la fois $\lambda$ et $-\lambda$ aussi bien que $g$ et $-g$ vont générer le même spectre asymptotique. Nous obtenons des expressions analytiques pour l'espacement entre les niveaux en fonction de $\lambda$ et $g$ pour des NPO généralisés.

[Traduit par la Rédaction]



## 1. Introduction

In quantum mechanics, expressing eigenspectra of the Schrödinger equation in an analytic form is very valuable as it provides analytic insight into the system. The analytic expressions for eigenvalues can only be obtained for a few potentials, such as the Coulomb potential, the Morse potential, the Poeschl–Teller potential, the square-well potential, and the harmonic oscillator.

On the other hand, rational potentials, such as the general nonpolynomial oscillators (NPOs) $V_N(x) = x^{2N} + (\lambda x^{m_1})/(1 + gx^{m_2})$, belong to a class of potentials where analytic eigenvalue expressions cannot be obtained exactly in general. These potentials are of importance in quantum field theory with a nonlinear Lagrangian, atomic physics and optical physics, as well as in elementary particle physics [1–10]. Apart from this, NPOs are themselves interesting as the real world deviates from an idealized picture of harmonic oscillators because of interactions between them and self interactions. As a result, a great deal of interest has been devoted to the investigation of one-dimensional NPOs, especially the potential $V(x) = x^2 + (\lambda x^2)/(1 + gx^2)$ in the past. However, the Schrödinger equation for this NPO cannot be solvable exactly for arbitrary values of $\lambda$, and $g$. Only in a few cases was it possible to obtain exact analytical solutions [11–16].

A wide variety of approaches, such as variational techniques, the Pade approximation method, the finite difference method, perturbation schemes, continued fractions method, and expansions into complete sets have been applied to the NPO $V(x) = x^2 + (\lambda x^2)/(1 + gx^2)$. Additionally, the concept of supersymmetric quantum mechanics has also been applied to such NPOs [17].

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The asymptotic energy expansion (AEE) method [18] was developed for potentials of the type $V(x) = \Sigma_{k=1}^{M} v_k(x)$ where $v_k(x)$ satisfy the scaling property $v_k(\lambda x) = \lambda^{n_k} v_k(x)$ for real $\lambda$ and integer $n_k$. This method produces analytic expressions for quantum action variable $J$ as an analytic series of energy $E$. Therefore the AEE method has been applied to investigate asymptotic behavior of eigenvalues (large eigenvalues) of polynomial potentials [19, 20] analytically. In addition, the AEE method has also been used to obtain the locations of zeros of wave functions of polynomial potentials [21, 22]. Because NPO $V(N, x)$ does not satisfy the scaling property mentioned earlier, the AEE method cannot be applied directly to NPOs.

In this paper the AEE method is extended to obtain a new type of recurrence relations, which can be used to obtain the AEE of NPOs ($V_N(x)$) in analytic form, having parameters $\lambda$ and $g$ explicitly in the series. We will also investigate the asymptotic behavior of eigenvalues and level spacings in terms of $\lambda$ and $g$ of NPOs.

In Sect. 2, the AEE method is extended and recurrence relations for NPOs $V_N(x)$ are derived. AEEs are obtained for few NPO potentials as illustrated in Sect. 3. In Sect. 4 asymptotic eigenenergies and level spacings of energy are obtained for general NPOs and concluding remarks are made in Sect. 5.

## 2. Extended AEE method for NPOs

In this section we will extend the AEE method and derive recurrence relations for the general NPO one-dimensional Hamiltonian

$$H(x, p) = p^2 + V(x, N) \tag{1}$$

where

$$V(N, x) = x^{2N} + \frac{\lambda x^{m_1}}{1 + g x^{m_2}}$$

where $N$, $m_1$, and $m_2$ are integers and $2N + m_2 > m_1$; $\lambda$ and $g$ are real parameters. These equations are required to obtain asymptotic semiclassical energy expansions in analytic form.

As in the polynomial case [20], the AEE quantization condition for NPOs is

$$J(E) = n\hbar \tag{2}$$

where $N$ is a positive integer and quantum action variable $J(E)$ is given by

$$J(E) = \frac{1}{2\pi}\int_{\gamma} P(x, E)\mathrm{d}x \tag{3}$$

$P(x, E)$ satisfies the equation

$$\frac{\hbar}{i}\frac{\partial P(x, E)}{\partial x} + P^2(x, E) = E - V(N, x) = P_{\mathrm{c}}(x, E) \tag{4}$$

and is related to the wave function as

$$P(x, E) = \frac{\hbar}{i}\frac{\partial \Psi/\partial x}{\Psi}$$

The contour $\gamma$ in (3) encloses two physical turning points of $P_{\mathrm{c}}(x, E)$. To obtain the AEE, first $P(x, E)$ is expanded in a series of powers of energy and subsequently obtain recurrence relations. For the preceding potential, (4) reads

$$\frac{\hbar}{i}\frac{\partial P(x, E)}{\partial x} + P^2(x, E) = E - x^{2N} - \frac{\lambda x^{m_1}}{1 + g x^{m_2}} \tag{5}$$

and has to be written in the form

$$\frac{\hbar}{i}\frac{\partial P}{\partial x} + P^2 + \frac{\hbar g x^{m_2}}{i}\frac{\partial P}{\partial x} + g x^{m_2} P^2 = E - x^{2N} + E g x^{m_2} - g x^{2N+m_2} - \lambda x^{m_1} \tag{6}$$

before deriving recurrence relations.

Let $\epsilon = E^{-1/2N}$ and $y = \epsilon x$. Then (6) becomes, after simplification

$$\frac{\hbar}{i}\frac{\epsilon^{2N+m_2+1}\partial P(y, \epsilon)}{\partial y} + \epsilon^{2N+m_2}P^2(y, \epsilon) + \frac{\hbar g y^{m_2}}{i}\frac{\epsilon^{2N+1}\partial P(y, \epsilon)}{\partial y} + g y^{m_2}\epsilon^{2N}P^2(y, \epsilon) = \left(1 - y^{2N}\right)\epsilon^{m_2} + g y^{m_2}\left(1 - y^{2N}\right) - \lambda y^{m_1}\epsilon^{2N+m_2-m_1} \tag{7}$$

Now we expand $P(y, \epsilon)$ as a power series in $\epsilon$.

$$P(y, \epsilon) = \epsilon^{s}\sum_{k=0}^{\infty} a_k(y)\epsilon^k \tag{8}$$

where $a_k$ and $k$ are determined later. Substituting (8) into (7) and equating coefficients of $\epsilon^0$, we obtain $s = -N$ and $a_0 = \sqrt{1 - y^{2N}}$ and (7) becomes

$$\frac{\hbar}{i}\sum_{k=0}^{\infty}\epsilon^{N+m_2+k+1}\frac{\mathrm{d}a_k}{\mathrm{d}y} + \frac{\hbar g y^{m_2}}{i}\sum_{k=0}^{\infty}\epsilon^{N+k+1}\frac{\mathrm{d}a_k}{\mathrm{d}y} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_i a_j \epsilon^{i+j+m_2} + g y^{m_2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_i a_j \epsilon^{i+j} = g y^{m_2}\left(1 - y^{2N}\right) + \left(1 - y^{2N}\right)\epsilon^{m_2} - \lambda y^{m_1}\epsilon^{2N+m_2-m_1} \tag{9}$$

and assuming $a_k = 0$ when $k < 0$ and rearranging terms, we obtain

$$\left(\frac{\hbar}{i}\sum_{k=1}^{\infty}\frac{\mathrm{d}a_{k-N-m_2-1}}{\mathrm{d}y} + \frac{\hbar g y^{m_2}}{i}\sum_{k=1}^{\infty}\frac{\mathrm{d}a_{k-N-1}}{\mathrm{d}y} + \sum_{k=1}^{\infty}\sum_{i=0}^{k-m_2}a_i a_{k-i-m_2} + g y^{m_2}\sum_{k=1}^{\infty}\sum_{i=1}^{k-1}a_i a_{k-i} + 2 g y^{m_2} a_0 a_k\right)\epsilon^k = g y^{m_2}\left(1 - y^{2N}\right) + \left(1 - y^{2N}\right)\epsilon^{m_2} - \lambda y^{m_1}\epsilon^{2N+m_2-m_1} \tag{10}$$

Then coefficients $a_k$ are given by

$$a_k = \frac{-1}{2 g y^{m_2} a_0}\left[\frac{\hbar}{i}\frac{\mathrm{d}a_{k-N-m_2-1}}{\mathrm{d}y} + \frac{\hbar g y^{m_2}}{i}\frac{\mathrm{d}a_{k-N-1}}{\mathrm{d}y} + (1 - \delta_{k,m_2})\sum_{i=0}^{k-m_2}a_i a_{k-i-m_2} + g y^{m_2}\sum_{i=1}^{k-1}a_i a_{k-i} + \lambda y^{m_1}\delta_{k,2N+m_2-m_1}\epsilon^{2N+m_2-m_1}\right] \tag{11}$$

where $a_k = 0 \;\forall\; k < 0$. Now $J$ can be written as

$$J(E) = \sum_{k=0}^{\infty} b_k E^{-(k-N-1)/2N} \quad (12)$$

where $b_k = (1/2\pi)\int_\gamma a_k \mathrm{d}y$ and can be determined analytically in terms of $\lambda$ and $g$. The contour $\gamma$ encloses the two branch points of $\sqrt{1-y^{2N}}$ (i.e., +1 and –1) on the real axis. The quantization condition $J(E) = n\hbar$ determines the eigenenergies of

$$V(N,x) = x^{2N} + \frac{\lambda x^{m_1}}{1+gx^{m_2}}$$

For polynomial potentials all the integrals $\int_\gamma a_k \mathrm{d}y$ have the general form either $\int[x^n/(1-x^{2N})^{m+1/2}]\mathrm{d}x$ or $\int[x^n/(1-x^{2N})^m]\mathrm{d}x$ where $m$ is an integer and $N$ is a positive integer [15]. However, the second integral, in general, does not contribute to $J$ except when $m = 1$ and $n = 2N - 1$.

On the other hand, for NPO potentials, in addition to the preceding two forms, another integral of the form $\int[1/x^l(1-x^{2N})^{m+1/2}]\mathrm{d}x$, with $l$ as a positive integer, contributes to the $J$. Importantly, all of the preceding integral forms can be evaluated analytically in terms of gamma functions.

## 3. Illustrations

In this section we present three simple illustrations for NPOs. The method described in this paper can be applied to a general NPO of the form $x^{2N} + \lambda x^{m_1}/(1+gx^{m_2})$ when $2N + m_2 > m_1$. Because AEE series contains terms that have positive powers of $\lambda$ in the numerator and $g$ in the denominator, the AEE method is accurate when $\lambda$ is small ($|\lambda| < |g|$) and $g$ is large ($|g| > 1$). The AEE expansion is more accurate for large eigenvalues and therefore, it is suitable for investigating asymptotic behavior of eigenvalues. It is convenient to use a computer algebra package such as MATHEMATICA to derive the AEE series. Using MATHEMATICA 8.0 [23], recurrence relation (10) was implemented with $a_0 = \sqrt{1-y^{2N}}$ for the three potentials. Results are given in the following subsections.

### 3.1. $V(x) = x^4 + \lambda x^2/(1 + gx^2)$

For this potential, (12) becomes

$$J(E) = \sum_{k=0}^{\infty} b_k E^{-(k-3)/4} \quad (13)$$

It is found that $b_k = 0\ \forall$ odd $k$ except $k = 3$ and $b_2$ is zero. The first six nonzero $b_k$s are

$$b_0 = \frac{\Gamma[1/4]}{3\sqrt{\pi}\Gamma[3/4]} \qquad b_3 = -\frac{\hbar}{2}$$

$$b_4 = -\left(\frac{\lambda}{g}\right)\frac{\Gamma[1/4]}{3\sqrt{\pi}\Gamma[3/4]}$$

$$b_6 = -\left(g^2\hbar^2 + 4\lambda\right)\frac{\Gamma[3/4]}{4g^2\sqrt{\pi}\Gamma[1/4]}$$

$$b_8 = -\lambda(8 + 3g\lambda)\frac{\Gamma[1/4]}{96g^3\sqrt{\pi}\Gamma[3/4]}$$

$$b_{10} = -3\lambda\left(16 + 5g^3\hbar^2 + 20g\lambda\right)\frac{\Gamma[3/4]}{80g^4\sqrt{\pi}\Gamma[1/4]}$$

**Table 1.** Some eigenvalues calculated with AEE for $V(x) = x^4 + [(\lambda x^2)/1 + gx^2]$ are compared with numerical eigenvalues. The calculation was carried out for $\lambda = 0.1$ with $g = 10$ and $\lambda = 1.0$ with $g = 50$ results are shown up to seven significant figures.

| | AEE | Numerical | AEE | Numerical |
|---|---|---|---|---|
| $N$ | $\lambda = 0.1$ | $g = 10$ | $\lambda = 1.0$ | $g = 50$ |
| 0 | 1.016 14 | 1.065 94 | 1.000 9 | 1.075 612 |
| 5 | 21.248 60 | 21.246 81 | 21.258 47 | 21.257 26 |
| 10 | 50.266 34 | 50.264 88 | 50.276 39 | 50.274 83 |
| 15 | 84.467 52 | 84.466 27 | 84.477 49 | 84.476 45 |
| 20 | 122.614 7 | 122.613 5 | 122.624 6 | 122.623 6 |
| 40 | 303.922 1 | 303.921 2 | 303.932 0 | 303.931 2 |
| 60 | 518.981 2 | 518.980 4 | 518.991 2 | 518.990 4 |
| 100 | 1021.000 | 1020.999 | 1021.009 | 1021.009 |
| 200 | 2564.207 | 2564.207 | 2564.217 | 2564.217 |

Applying quantization condition $J(E) = n\hbar$, the eigenenergies are obtained. For comparison purposes, the Schrödinger equation for the preceding potential was solved numerically as well. Eigenenergies are shown in Table 1.

Although the eigenvalues obtained from the numerical method were not exact, here we assume that they are at least accurate up to seven significant figures. It is apparent from Table 1 that the AEE method produced less accurate eigenvalues when $N$ is small. However, as $N$ increases, an increase in the accuracy of the AEE method has become evident. The analytic expression of $J$ presented in (13) for the preceding potential can be utilized to investigate the asymptotic behavior of eigenvalues analytically.

### 3.2. $V(x) = x^4 + \lambda x^2/(1 + gx)$

The next illustration is for $V(x) = x^4 + \lambda x^2/(1 + gx)$ and the AEE expansion is

$$J(E) = \sum_{k=0}^{\infty} b_k E^{-(k-3)/4} \quad (14)$$

As in the previous illustration, $b_k = 0\ \forall$ odd $k$ except $k = 3$ and $b_2 = 0$. The first six nonzero $b_k$s are

$$b_0 = \frac{\Gamma[1/4]}{3\sqrt{\pi}\Gamma[3/4]} \qquad b_3 = -\frac{\hbar}{2} \qquad b_4 = \frac{\lambda\Gamma[1/4]}{4g^2\sqrt{\pi}\Gamma[3/4]}$$

$$b_6 = -\left(2g^4\hbar^2 + 8\lambda - g^2\lambda^2\right)\frac{\Gamma[3/4]}{8g^4\sqrt{\pi}\Gamma[1/4]}$$

$$b_8 = -\lambda\left(-8 + 9g^2\lambda\right)\frac{\Gamma[1/4]}{96g^6\sqrt{\pi}\Gamma[3/4]}$$

$$b_{10} = 3\lambda\left(-32 + 10g^6\hbar^2 + 100g^2\lambda - 5g^4\lambda^2\right)\frac{\Gamma[3/4]}{160g^8\sqrt{\pi}\Gamma[1/4]}$$

For this system the AEE energies are given in Table 2 along with numerically calculated energies. As in the first illustration, the eigenvalues calculated with the AEE method became more accurate for large $N$.

**Table 2.** Some eigenvalues calculated with the AEE method for $V(x) = x^4 + [(\lambda x^2)/1 + gx]$ are compared with numerical eigenvalues. The calculation was carried out for $\lambda = 1$ with $g = 10$ and $\lambda = 5$ with $g = 50$ and results are shown up to seven significant figures.

| | AEE | Numerical | AEE | Numerical |
|---|---|---|---|---|
| $N$ | $\lambda = 1.0$ | $g = 10$ | $\lambda = 5.0$ | $g = 50$ |
| 0 | 0.971 111 5 | 1.052 777 | 0.978 339 | 1.057 186 |
| 5 | 21.228 33 | 21.227 72 | 21.236 32 | 21.236 24 |
| 10 | 50.246 19 | 50.246 53 | 50.254 18 | 50.254 17 |
| 15 | 84.447 41 | 84.446 79 | 84.455 41 | 84.455 40 |
| 20 | 122.594 5 | 122.594 9 | 122.602 6 | 122.602 6 |
| 40 | 303.902 0 | 303.902 0 | 303.910 0 | 303.910 0 |
| 60 | 518.961 1 | 518.960 9 | 518.969 2 | 518.969 2 |
| 100 | 1020.979 | 1020.981 | 1020.988 | 1020.988 |
| 200 | 2564.187 | 2564.187 | 2564.195 | 2564.195 |

### 3.3. $V(x) = x^8 + \lambda x^2/(1 + gx)$

The AEE expansion for $V(x) = x^8 + \lambda x^2/(1 + gx)$ is

$$J(E) = \sum_{k=0}^{\infty} b_k E^{-(k-5)/8} \tag{15}$$

It is found that $b_k = 0$ ∀ odd $k$ except $k = 5$ and $b_2$, $b_4$, and $b_6$ are all zero. The first six nonzero $b_k$s are

$$b_0 = \frac{\Gamma[1/8]}{5\sqrt{\pi}\Gamma[5/8]} \qquad b_5 = -\frac{\hbar}{2}$$

$$b_8 = -\frac{\lambda\Gamma[1/8]}{8g^2\sqrt{\pi}\Gamma[5/8]}$$

$$b_{10} = -\left(7g^4\hbar^2 + 12\lambda\right)\frac{\Gamma[7/8]}{12g^4\sqrt{\pi}\Gamma[3/8]}$$

$$b_{12} = \left(5g^4\hbar^2 - 4\lambda\right)\frac{\Gamma[5/8]}{12g^6\sqrt{\pi}\Gamma[1/8]}$$

$$b_{14} = -\lambda\left(-16 + 5g^6\lambda\right)\frac{\Gamma[3/8]}{640g^8\sqrt{\pi}\Gamma[7/8]}$$

In Table 3, the AEE energies and numerically calculated energies are given for comparison. Very low eigenvalues do not agree with the numerical eigenvalues. However, when $n > 15$ the eigenvalues agree well with the numerically calculated values as shown earlier.

## 4. Asymptotic behavior and energy level spacings for general potential $V(x) = x^{2N} + [(\lambda x^{m_1})/(1 + gx^{m_2})]$

First, the recurrence relations in (10) is utilized in this section to obtain four terms of the AEE series analytically for the general NPO potential $V(x) = x^{2N} + (\lambda x^{m_1})/(1 + gx^{m_2})$. These terms will be adequate to study the asymptotic behavior of eigenvalues and the lever spacings. Even with a computer algebra package, the $b_k$ terms cannot be derived directly when $N$, $m_1$, and $m_2$ are not assigned numerical values. However, by inspecting terms in the AEE series systematically for various values of $N$, $m_1$, and $m_2$, one can generalize the $b_k$ terms of AEE for the general NPO potential. It was found that there are four different forms of AEE expansions for the above potential corresponding to parameters $m_1$ and $m_2$ being even or odd. These four forms arise when (*i*) $(m_1 - m_2)$ is even; (*ii*) both $m_1$ and $m_2$ are odd; (*iii*) $m_1$ is odd while $m_2$ is even but $N \neq 2(m_1 - m_2) + 1$; and (*iv*) $m_1$ is odd while $m_2$ is even but $N = 2(m_1 - m_2) + 1$.

Equations (16)–(32) provide the AEE expansion of the preceding general potential with four terms.

1. If $(m_1 - m_2)$ is even, the AEE is

$$J(E) = b_0 E^{(N+1)/2N} + b_{N+1} + b_{2N+2}E^{-(N+1)/2N} + b_{2N+m_2-m_1}E^{-(N+m_2-m_1-1)/2N} \tag{16}$$

where

$$b_0 = \frac{\Gamma[1/2N]}{(N+1)\sqrt{\pi}\Gamma[(1/2) + (1/2N)]} \tag{17}$$

$$b_{N+1} = -\frac{\hbar}{2} \tag{18}$$

$$b_{2N+2} = -\frac{(2N-1)\hbar^2\Gamma[(2N-1)/2N]}{12\sqrt{\pi}\Gamma[(N-1)/2N]} \tag{19}$$

if $N < m_1 - m_2 + 1$

$$b_{2N+m_2-m_1} = -\left(\frac{\lambda}{g}\right)\frac{\Gamma[(m_1 - m_2 + 1)/2N]}{(m_1 - m_2 - N + 1)\sqrt{\pi}\Gamma[(N + m_1 - m_2 + 1)/2N]} \tag{20}$$

if $N = m_1 - m_2 + 1$

$$b_{2N+m_2-m_1} = -\left(\frac{\lambda}{2Ng}\right) \tag{21}$$

if $N > m_1 - m_2 + 1$

$$b_{2N+m_2-m_1} = -\left(\frac{\lambda}{g}\right)\frac{\Gamma[(m_1 - m_2 + 1)/2N]}{2N\sqrt{\pi}\Gamma[(N + m_1 - m_2 + 1)/2N]} \tag{22}$$

**Table 3.** Some eigenvalues calculated with the AEE method for the potential $V(x) = x^8 + [(\lambda x^4)/1 + gx]$ are compared with numerical eigenvalues. The calculation was carried out for $\lambda = 1$ with $g = 10$ and $\lambda = 5$ with $g = 50$.

| | AEE | Numerical | AEE | Numerical |
|---|---|---|---|---|
| $N$ | $\lambda = 1.0$ | $g = 10$ | $\lambda = 5.0$ | $g = 50$ |
| 0 | 1.127 239 | 1.222 547 | 1.130 126 3 | 1.224 879 |
| 5 | 35.487 08 | 35.487 48 | 35.495 239 | 35.495 693 |
| 10 | 99.534 93 | 99.534 99 | 99.545 484 | 99.545 595 |
| 15 | 185.489 7 | 185.489 8 | 185.502 07 | 185.502 11 |
| 20 | 290.064 6 | 290.064 6 | 290.078 42 | 290.078 43 |
| 40 | 862.027 1 | 862.027 1 | 862.045 19 | 862.045 19 |
| 60 | 1 638.267 9 | 1 638.267 9 | 1 638.289 03 | 1 638.289 03 |
| 100 | 3 690.067 99 | 3 690.067 99 | 3 690.093 92 | 3 690.093 92 |
| 200 | 11 141.642 23 | 11 141.642 23 | 11 141.676 37 | 11 141.676 37 |



2. If $(m_1 - m_2)$ is odd and $m_2$ is odd, the AEE is

$$J(E) = b_0 E^{(N+1)/2N} + b_{N+1} + b_{2N+2} E^{-(N+1)/2N} + b_{2N-m_1+2m_2} E^{-(N+2m_2-m_1-1)/2N} \tag{23}$$

where $b_0$, $b_{N+1}$, and $b_{2N+2}$ are the same as when $(m_1 - m_2)$ is even and if $m_1 < 2(m_2 - 1) - N$

$$b_{2N+2m_2-m_1} = \left(\frac{\lambda}{g^2}\right)\frac{(2m_2 - m_1 - N - 1)\Gamma[(2N + m_1 - 2m_2 + 1)/2N]}{2N(2m_2 - m_1 - 1)\sqrt{\pi}\Gamma[(3N + m_1 - 2m_2 + 1)/2N]} \tag{24}$$

if $2(m_2 - 1) - N < m_1 < 2(m_2 - 1)$

$$b_{2N+2m_2-m_1} = -\left(\frac{\lambda}{g^2}\right)\frac{\Gamma[(2N + m_1 - 2m_2 + 1)/2N]}{(2m_2 - m_1 - 1)\sqrt{\pi}\Gamma[(N + m_1 - 2m_2 + 1)/2N]} \tag{25}$$

if $m_1 = 2(m_2 - 1)$

$$b_{2N+2m_2-m_1} = -\left(\frac{\lambda}{g^2}\right)\frac{\Gamma[(2N + m_1 - 2m_2 + 1)/2N]}{\sqrt{\pi}\Gamma[(N + m_1 - 2m_2 + 1)/2N]} \tag{26}$$

if $m_1 > 2(m_2 - 1)$

$$b_{2N+2m_2-m_1} = \left(\frac{\lambda}{g^2}\right)\frac{\Gamma[(m_1 - 2m_2 + 1)/2N]}{2N\sqrt{\pi}\Gamma[(N + m_1 - 2m_2 + 1)/2N]} \tag{27}$$

3. If $(m_1 - m_2)$ is odd while $m_2$ is even and $N \neq 2(m_1 - m_2) + 1$ the AEE is

$$J(E) = b_0 E^{(N+1)/2N} + b_{N+1} + b_{2N+2} E^{-(N+1)/2N} + b_{4N+2m_2-2m_1} E^{-(3N+2m_2-2m_1-1)/2N} \tag{28}$$

where $b_0$, $b_{N+1}$, and $b_{2N+2}$ are the same as when $(m_1 - m_2)$ is even and if $N < 2(m_1 - m_2) + 1$

$$b_{4N+2m_2-2m_1} = -\left(\frac{\lambda^2}{g^2}\right)\frac{(N - 2m_1 + 2m_2 - 1)\{\Gamma[2(m_1 - m_2) + 1]/2N\}}{8N^2\sqrt{\pi}\Gamma\{[N + 2(m_1 - m_2) + 1]/2N\}} \tag{29}$$

if $N > 2(m_1 - m_2) + 1$

$$b_{4N+2m_2-2m_1} = -\left(\frac{\lambda^2}{g^2}\right)\frac{\Gamma\{[2(m_1 - m_2) + 1]/2N\}}{4N\sqrt{\pi}\Gamma\{[N + 2(m_1 - m_2) + 1]/2N\}} \tag{30}$$

4. If $(m_1 - m_2)$ is odd and $m_2$ is even with $N = 2(m_1 - m_2) + 1$ the AEE is

$$J(E) = b_0E^{(N+1)/2N} + b_{N+1} + b_{2N+2}E^{-(N+1)/2N} + b_{3N+m_2}E^{-(2N+m_2-1)/2N} \tag{31}$$

where $b_0$, $b_{N+1}$, and $b_{2N+2}$ are the same as when $(m_1 - m_2)$ is even and

$$b_{3N+m_2} = -\left(\frac{\lambda^2}{g^3}\right)\frac{m_2\Gamma[(N - m_2)/2N]}{4N^2\sqrt{\pi}\Gamma[(2N - m_2)/2N]} \tag{32}$$

Note that $b_{2N+m_2-m_1}$, $b_{2N+2m_2-m_1}$, $b_{4N+2m_2-2m_1}$, and $b_{3N+m_2}$ are the lowest order terms that contain parameters λ and $g$ in cases 1, 2, 3, and 4, respectively. Consequently, for asymptotic energies, these terms will supply the largest contribution to λ and $g$.

Having obtained four coefficients of AEE, now we investigate how the eigenvalues behave asymptotically. Four terms in each of these AEE expansions can be used to investigate how the asymptotic eigenenergies will depend on the parameters $N$, $m_1$, $m_2$, λ, and $g$. Because the fourth term in each of the AEE expansions is the most dominating term having $m_1$, $m_2$, λ, and $g$, it can be used to investigate how the parameters of the potential will contribute to asymptotic eigenenergy spectra. It is interesting to note that for all the possible values of $N$, $m_1$, and $m_2$, the contribution of λ and $g$ are only in the form λ or $\lambda^2$, and $g^{-1}$, $g^{-2}$, or $g^{-3}$.

Because (20), (22), (29), and (30) contain $m_1$ and $m_2$ in the form $m_1 - m_2$, all the $m_1$ and $m_2$ having the same difference $m_1 - m_2$ will produce almost the same asymptotic eigenenergy spectra in each case. Similarly, it is evident from (24)–(27) that when $m_1$ is even while $m_2$ is odd, the same $m_1 - 2m_2$ will generate almost the same asymptotic eigenspectra for each case separately.

When $m_1 - m_2$ is odd, $m_2$ is even and $N \neq 2(m_1 - m_2) + 1$, both λ and –λ generate almost the same asymptotic eigenenergies as terms (29) and (30), which contain λ only in the form of $\lambda^2$. Similarly, according to (24)–(30), when $m_1 - m_2$ is odd, $g$ and $-g$ will produce almost the same asymptotic eigenenergy spectra except for $N = 2(m_1 - m_2) + 1$.

For large $N$, $b_0 = 2/\pi$, $b_{N+1} = -(\hbar/2)$, and $b_{2N+2} = -[(2N - 1)\hbar^2/12\pi]$, For large energies, the energy level spacing is approximately given by

$$\Delta E \approx \frac{1}{(\partial J/\partial E)} \tag{33}$$

First we write the AEE as

$$J(E) = b_0E^{(N+1)/2N} + b_{N+1} + b_{2N+2}E^{-(N+1)/2N} + b_\alpha E^{-(\alpha-N-1)/2N} \tag{34}$$

where $\alpha$ takes value $2N + m_2 - m_1$, $2N - m_1 + 2m_2$, $4N + 2m_2 - 2m_1$, or $3N + m_2$ depending on cases as described earlier. Now the level spacing $\Delta E$ becomes

$$\Delta E = c_0E^{(N-1)/2N} + c_1E^{-(N+3)/2N} + c_\alpha E^{-(\alpha-N+1)/2N} \tag{35}$$

where $c_0 = 2N/(N + 1)b_0$, $c_1 = [2Nb_{2N+2}/(N + 1)]c_0^2$, and $c_\alpha = [(\alpha - N + 1)b_\alpha/2N]c_0^2$.

For large $N$, $c_0 \approx \pi$, $c_1 \approx -(1/6)(2N - 1)\hbar^2\pi$, and $c_\alpha \approx -(\lambda/2g)[\pi/(m_1 - m_2 + 1)]$ when $(m_1 - m_2)$ is even; $c_\alpha \approx -(\lambda/g^2)[\pi/(m_1 - 2m_2 + 1)]$ when both $(m_1 - m_2)$ and $m_2$ are odd; and $c_\alpha \approx -(\lambda^2/g^2)[3\pi/4(2m_1 - 2m_2 + 1)]$ when $(m_1 - m_2)$ is odd and $m_2$ is even.

Here we have employed the relation $\Gamma[1/x] \to x - \gamma$ for large $x$, where $\gamma$ is the Euler constant.

When $(m_1 - m_2)$ is odd and $m_2$ is even

$$\Delta E = \pi E^{1/2} + c_1E^{-1/2} + c_\alpha E^{-3/2} \tag{36}$$

or

$$\Delta E = \pi E^{1/2} + (c_1 + c_\alpha)E^{-1/2} \tag{37}$$

otherwise.

Therefore, very large $N$, asymptotic eigenvalues have the level spacing

$$\Delta E \sim \pi\sqrt{E} \tag{38}$$

regardless of the values of λ, $g$, $m_1$, and $m_2$.

Now we investigate how the asymptotic eigenenergies change with parameters λ and $g$. The first-order approximation to the change in energy $\Delta E_\lambda$ due to changes in λ is given by

$$\Delta E_\lambda \approx \frac{(\partial J/\partial\lambda)}{(\partial J/\partial E)}\Delta\lambda \tag{39}$$

Because only the fourth term contains the parameter λ

$$\Delta E_\lambda \approx \frac{\partial b_\alpha}{\partial\lambda}\left(\frac{2N}{b_0(N + 1)}\right)E^{-[(\alpha-2N)/2N]}\Delta\lambda \tag{40}$$

and similarly changes in energy $\Delta E_g$ due to changes in $g$ are given by

$$\Delta E_g \approx \frac{\partial b_\alpha}{\partial g}\left(\frac{2N}{b_0(N + 1)}\right)E^{-[(\alpha-2N)/2N]}\Delta g \tag{41}$$

Now we define $d_0$, $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, and $d_6$ as

**Table 4.** Changes of asymptotic eigen energies with respect to changes in $\lambda$ and $g$.

| $m_1 - m_2$ | Condition | $\Delta E_\lambda$ | $\Delta E_g$ |
|---|---|---|---|
| — | $N<m_1 - m_2 + 1$ | $-(1/g)d_0\Delta\lambda$ | $(\lambda/g^2)d_0\Delta g$ |
| $m_1 - m_2$ is *even* | $N = m_1 - m_2 + 1$ | $-(1/g)d_1\Delta\lambda$ | $(\lambda/g^2)d_1\Delta\lambda\Delta g$ |
| — | $N > m_1 - m_2 + 1$ | $-(1/g)d_2\Delta\lambda$ | $(\lambda/g^2)d_2\Delta g$ |
| — | $m_1 < 2(m_2 - 1) - N$ | $-(1/g^2)d_3\Delta\lambda$ | $(2\lambda/g^3)d_3\Delta g$ |
| $m_1 - m_2$ is *odd* | $2(m_2 - 1) - N < m_1 < 2(m_2 - 1)$ | $-(1/g^2)d_4\Delta\lambda$ | $(2\lambda/g^3)d_4\Delta g$ |
| and $m_2$ is odd | $m_1 = 2(m_2 - 1)$ | $-(1/g^2)d_5\Delta\lambda$ | $(2\lambda/g^3)d_5\Delta g$ |
| — | $m_1 > 2(m_2 - 1)$ | $-(1/g^2)d_6\Delta\lambda$ | $-(2\lambda/g^3)d_6\Delta g$ |
| $m_1 - m_2$ is *odd* | $N < 2(m_1 - m_2) + 1$ | $-(2\lambda/g^2)d_7\Delta\lambda$ | $(2\lambda^2/g^3)d_7\Delta g$ |
| and $m_2$ is even | $N = 2(m_1 - m_2) + 1$ | $-(2\lambda/g^3)d_8\Delta\lambda$ | $(3\lambda^2/g^4)d_8\Delta g$ |
| — | $N > 2(m_1 - m_2) + 1$ | $-(2\lambda/g^2)d_9\Delta\lambda$ | $(2\lambda^2/g^3)d_9\Delta g$ |

$$
\begin{aligned}
d_0 &= \frac{2N\Gamma[(m_1-m_2+1)/2N]}{(N+1)(m_1-m_2-N+1)b_0\sqrt{\pi}\Gamma[(N+m_1-m_2+1)/2N]}E^{-(m_2-m_1/2N)}\\
d_1 &= -\left(\frac{1}{(N+1)b_0}\right)E^{-(m_2-m_1)/2N}\\
d_2 &= \frac{\Gamma[(m_1-m_2+1)/2N]}{(N+1)b_0\sqrt{\pi}\Gamma[(N+m_1-m_2+1)/2N]}E^{-(m_2-m_1)/2N}\\
d_3 &= \frac{(2m_2-m_1-N-1)\Gamma[(2N+m_1-2m_2+1)/2N]}{(N+1)(2m_2-m_1-1)b_0\sqrt{\pi}\Gamma[(3N+m_1-2m_2+1)/2N]}E^{-(2m_2-m_1)/2N}\\
d_4 &= \frac{2N\Gamma[(2N+m_1-2m_2+1)/2N]}{(N+1)(2m_2-m_1-1)b_0\sqrt{\pi}\Gamma[(N+m_1-2m_2+1)/2N]}E^{-(2m_2-m_1)/2N}\\
d_5 &= \frac{2N\Gamma[(2N+m_1-2m_2+1)/2N]}{(N+1)b_0\sqrt{\pi}\Gamma[(N+m_1-2m_2+1)/2N]}E^{-(2m_2-m_1)/2N}\\
d_6 &= \frac{\Gamma[(m_1-2m_2+1)/2N]}{(N+1)b_0\sqrt{\pi}\Gamma[(N+m_1-2m_2+1)/2N]}E^{-(2m_2-m_1)/2N}\\
d_7 &= \frac{(N-2m_1+2m_2-1)\Gamma[(2(m_1-m_2)+1)/2N]E^{-(3N+2m_2-2m_1)/2N}}{4N(N+1)b_0\sqrt{\pi}\Gamma\{[N+2(m_1-m_2)+1]/2N\}}\\
d_8 &= \frac{m_2\Gamma[(N-m_2)/2N]}{4N(N+1)b_0\sqrt{\pi}\Gamma[(2N-m_2)/2N]}E^{-(2N+m_2-1)/2N}\\
d_9 &= \frac{\Gamma[(2(m_1-m_2)+1)/2N]}{2(N+1)b_0\sqrt{\pi}\Gamma\{[N+2(m_1-m_2)+1]/2N\}}E^{-(3N+2m_2-2m_1)/2N}
\end{aligned} \tag{42}
$$

and express $\Delta E_\lambda$ and $\Delta E_g$ in terms of $d_1 \ldots d_6$ as shown in Table 4.

It is evident from Table 4 that as $\lambda$ increases (when $\lambda$ is positive and $g$ is positive), the asymptotic eigenenergies decrease for all values of $N$, $m_1$, and $m_2$ except for the case where $m_1$ and $m_2$ are odd with $m_1 < 2(m_2 - 1) - N$ or $m_1$ and $m_2$ are odd with $m_1 > 2(m_2 - 1)$. On the other hand, as $g$ increases (when $\lambda$ is positive and $g$ is positive), the asymptotic eigenenergies increase for all values of $N$, $m_1$, and $m_2$ except for the case where $m_1$ and $m_2$ are odd with $m_1 < 2(m_2 - 1) - N$ or $m_1$ and $m_2$ are odd with $m_1 > 2(m_2 - 1)$.

## 5. Concluding remarks

In this paper, we have derived AEEs for the NPO $x^{2N} + [\lambda x^{m_1}/(1 + gx^{m_2})]$. Because AEE has been developed only for polynomial potentials, it cannot directly be applied to nonpolynomial systems. In this paper new types of recurrence relations were derived for NPOs and applied to the preceding general NPO to obtain expressions for $J$ in terms of $E$ and the parameters of the potential. These expansions are power series in energy and coefficients of the series contain parameters $\lambda$ and $g$ explicitly. The most significant first four terms of the AEE were derived as analytic expressions explicitly in terms of $N$, $m_1$, $m_2$, $\lambda$, and $g$. Using these AEE expressions, it was shown that there are classes of NPOs with different $m_1$ and $m_2$ but with the same $m_1 - m_2$ or $m_1 - 2m_2$, share the same asymptotic eigenspectra if both have the same $N$. It was also shown that for certain cases, both $\lambda$ and $-\lambda$ as well as $g$ and $-g$ will produce the same asymptotic energy spectra. Therefore, extended AEE is a very useful tool in investigating NPOs analytically and they provide analytic insight into the problem.

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