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Explicit energy expansion for general odd-degree polynomial potentials

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Abstract

In this paper we derive an almost explicit analytic formula for asymptotic eigenenergy expansion of arbitrary odd-degree polynomial potentials of the form $V(x) = (ix)^{2N+1} + \beta_1 x^{2N} + \beta_2 x^{2N-1} + \dots + \beta_{2N} x$, where β'_k are real or complex for $1 \le k \le 2N$. The formula can be used to find semiclassical analytic expressions for eigenenergies up to any order, very efficiently. Each term of the expansion is given explicitly as a multinomial of the parameters $\beta_1, \beta_2 \dots$ and β_{2N} of the potential. Unlike in the even-degree polynomial case, the highest-order term in the potential is pure imaginary and hence the system is non-Hermitian. Therefore all the integrations have been carried out along a contour enclosing two complex branch points, which lies within a wedge in the complex plane. With the help of some examples we demonstrate the accuracy of the method for both real and complex eigenspectra.

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1. Introduction

A Hamiltonian with an odd-degree polynomial potential

$$H = \frac{p^2}{2} + \beta_0 x^{2N+1} + \beta_1 x^{2N} + \beta_2 x^{2N-1} + \dots + \beta_{2N} x \qquad (1)$$

is PT symmetric when $\beta_0, \beta_2, \ldots, \beta_{2N}$ are pure imaginary and $\beta_1, \beta_3, \ldots, \beta_{2N-1}$ are real. It has real eigen spectra when PT symmetry is not spontaneously broken (i.e. when the wave functions are also PT symmetric). Moreover, the wave functions of H are usually required to vanish at infinity in various Stokes wedges to satisfy boundary conditions for quantization. Therefore, solving the Schrödinger equation directly to obtain eigen spectra for these systems is not a trivial task. Recently the eigenenergy spectra of general polynomial potentials has been investigated using the spectral resolution method and extended Wentzel–Kramers–Brillouin (WKB) methods [1–12].

At present, WKB theory is well developed and its methods are very important for many applications. The WKB method has been extended to obtain higher-order eigenenergies for potentials such as $V(x) = x^{2N}$ [13]. For this system, the integrals in each term of the expansion

can be evaluated analytically in terms of Γ functions. Although, at first sight the problem of obtaining higher-order terms of WKB seems relatively simple for other systems, it has proved difficult due to singularities at the classical turning points [14]. However, the lowest-order WKB method has been applied to obtain eigenenergies of many PT symmetric potentials, such as $V(x) = gx^2(ix)^{\varepsilon}$. Robnik and Romanovski [14, 15] derived a simple formula for the semiclassical series for the potentials $V(x) = x^{2N}$ and obtained explicit formulae for the WKB approximation of the eigenenergies of the same. The recurrence relations obtained by Robnik et al are computationally much less time-consuming compared to the WKB recurrence relations when the order increases. The reason behind this difference is that Robnik's recurrence relations involve only arithmetic operations with rational numbers while WKB formulae involve operations of differentiation and a collection of similar terms. In addition, they have derived almost explicit formulae for the WKB terms of the energy eigenvalues of the potential $V(x) = x^{2N}$. However, use of the method developed by Robnik et al for general polynomial potentials is not possible due to the complicated nature of the integrals involved.

Compared to the higher-order WBK expansion, derivation of the asymptotic energy expansion (AEE) [16–19] for polynomial type potentials is relatively easy.

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All of the integrals involved in the AEE of polynomial potentials can be evaluated analytically in terms of Γ functions. As a result, using Robnik's method in conjunction with the AEE method, an almost explicit formula for semiclassical energies for even degree real polynomial potentials has been derived [6]. One of the important features of this formula is that it contains parameters $a_1, a_2, \ldots, a_{2N-1}$ of the potential $V(x) = x^{2N} + a_1 x^{2N-1} + a_2 x^{2N-2} + \cdots + a_{2N-1} x$, explicitly. Since AEE is an expansion in reciprocal of energy, energy is also present explicitly in the formula. Recently, the AEE method developed for even-degree polynomial potentials has been successfully applied for finding equivalent Hermitian Hamiltonians for non-Hermitian Hamiltonians [20].

In a recent paper, Bender and Jones [21] developed a technique based on WKB theory to obtain the behavior of eigenenergy levels of the potentials of the type V(x) = $-igx^{2N+1}$ (for integer N) as g varies. The method is accurate enough to determine the critical points where the pairs of real eigenvalues are merged and become complex conjugate pairs. Due to the complicated nature of the integrals, this WKB method cannot also be applied for general odd-degree polynomial potentials. Recently, the AEE method has been extended for the potential of the type $V(x) = \mu x^3 + ax^2 + bx$ [19]. Integrals in the AEE expansion of this system contain odd powers of $\sqrt{1-y^3}$, where y is the integration variable. Therefore, the integrals are evaluated with contours enclosing branch points 1 and ∞ . However, the extension of the AEE method to higher-order odd-degree polynomial potentials (degree > 3) failed to produce correct energy spectra. In this study it was found that the reason for this failure was the choice of branch points. In this paper we extend the AEE method to higher-order odd-degree polynomial potentials with the correct choice of branch points for degrees ≥ 3 and contours and derive an almost explicit analytic formula for asymptotic eigenenergy expansion for arbitrary odd-degree polynomial potentials of the form

$$V(x) = (ix)^{2N+1} + \beta_1 x^{2N} + \beta_2 x^{2N-1} + \dots + \beta_{2N} x, \qquad (2)$$

where *N* is a positive integer and $\beta_k \in \mathbb{C}$ for $1 \leq k \leq 2N$. The system has real eigenvalues when V(x) is *PT* symmetric and eigen spectrum is complex otherwise. The AEE expansion is valid for both real and complex asymptotic eigenenergies. The paper is organized as follows. In section 2 we extend the AEE method for odd-degree polynomial potentials with two terms and evaluate the accuracy of both real and complex eigenenergies. The main result of this paper, an almost explicit analytic formula for asymptotic eigenenergy expansion for the potential (2), is derived in section 3. Two examples are given in section 3 to demonstrate the accuracy of the AEE method as well as the derived formula. In section 4, a summary and concluding remarks are presented.

2. Analytic semiclassical energy expansions for $V(x) = (\mathbf{i}x)^{2N+1} + bx$

In this section we investigate the two term odd-degree polynomial potentials for the form

$$V(x) = (ix)^{2N+1} + bx,$$
(3)

where $b \in \mathbb{C}$. Consider the one dimensional Schrödinger equation

$$-\hbar^2 \frac{\partial^2 U(x, E)}{\partial x^2} + V(x) U(x, E) = EU(x, E).$$
(4)

Substituting $P(x, E) = \frac{\hbar}{i} \frac{\partial U(x, E)/\partial x}{U(x, E)}$ in the above equation, we get

$$\frac{\hbar}{i}\frac{\partial P(x,E)}{\partial x} + P^2(x,E) = E - V(x).$$
(5)

Note that P(x, E) above corresponds to the derivative of the action in the usual WKB ansatz. The quantity J(E) is now defined as

$$J(E) = \frac{1}{2\pi} \int_{\gamma} P(x, E) dx$$
 (6)

with the quantization condition $J(E) = n\hbar$. The contour γ encloses two turning points of $P_c = \sqrt{E - V(x)}$.

For the potential in equation (3), equation (5) becomes

$$\frac{h}{i}\frac{\partial P(x,E)}{\partial x} + P^2(x,E) = E - (ix)^{2N+1} - bx.$$
(7)

Let $\epsilon = E^{-\frac{1}{4N+2}}$ and $y = i\epsilon^2 x$. Then (7) becomes, after simplification,

$$\hbar \epsilon^{4N+4} \frac{\partial P(y,\epsilon)}{\partial y} + \epsilon^{4N+2} P^2(y,\epsilon) = 1 - y^{2N+1} - \frac{by}{i} \epsilon^{4N}.$$
(8)

Now we expand $P(y, \epsilon)$ as a power series in ϵ ,

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$$P(y,\epsilon) = \epsilon^{s} \sum_{n=0}^{\infty} a_{n}(y) \epsilon^{n}, \qquad (9)$$

where *s* and $a_n(y)$ are determined below. Substituting (9) in (8) and equating coefficients of ϵ^0 , we obtain s = -(2N + 1) and $a_0 = \sqrt{1 - y^{2N+1}}$ and (8) becomes

$$\hbar \sum_{n=0}^{\infty} \frac{\mathrm{d}a_n}{\mathrm{d}y} \epsilon^{2N+n+3} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i a_j \epsilon^{i+j} = 1 - y^{2N+1} - \frac{by}{\mathrm{i}} \epsilon^{4N},$$
(10)

and rearranging terms, we obtain

$$\left(\hbar \sum_{n=1}^{\infty} \frac{\mathrm{d}a_{n-2N-3}}{\mathrm{d}y} + \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} a_i a_{n-i} + 2a_0 \sum_{n=0}^{\infty} a_n\right) \times \epsilon^n = 1 - y^{2N+1} - \frac{by}{\mathrm{i}} \epsilon^{4N}.$$
(11)

Then coefficients a_n are given by

$$a_n = \frac{-1}{2a_0} \left[\sum_{i=1}^{n-1} a_i a_{n-i} + \hbar \frac{\mathrm{d}a_{n-2N-3}}{\mathrm{d}y} + \frac{by}{\mathrm{i}} \delta_{4N,n} \right].$$
(12)

In the above formula $a_n = 0 \forall n < 0$. The first four non-zero a_n for given N are

$$a_0 = \sqrt{1 - y^{2N+1}},$$

$$a_{2N+3} = -\frac{\hbar}{2a_0} \frac{\mathrm{d}a_0}{\mathrm{d}y}$$

$$a_{4N} = -\frac{by}{2ia_0},$$

$$a_{4N+6} = -\frac{1}{2a_0} \left[a_{2N+3}^2 + \hbar \frac{\mathrm{d}a_{2N+3}}{\mathrm{d}y} \right]$$

Now J can be written as

$$J(E) = \sum_{k=0}^{\infty} b_k E^{\frac{-(k-2N-3)}{4N+2}},$$
(13)

where

$$b_k = \frac{1}{2i\pi} \int\limits_{\gamma} a_k(y) \, \mathrm{d}y. \tag{14}$$

Now, in order to evaluate the integral (14), the contour γ is chosen such that it encloses the two branch points of $\sqrt{1-y^{2N+1}}$ on the complex plane. There are 2N + 1 branch points on the complex plane. The branch points which should be enclosed by the contour γ are $(-1)^N e^{iN\pi/(2N+1)}$ and $(-1)^{N+1} e^{i(N+1)\pi/(2N+1)}$. Note that these two branch points lie inside the Stokes wedges which are necessary for defining the above non-Hermitian problem correctly as an eigenvalue problem [7].

The integration is then carried out for each term and obtained the expression for J(E) as

$$J(E) = b_0 E^{\frac{2N+3}{4N+2}} + b_{2N+3} + b_{4N} E^{-\frac{2N-3}{4N+2}} + b_{4N+6} E^{-\frac{2N+3}{4N+2}}, \quad (15)$$

where

$$b_0 = \frac{2\cos[\frac{\pi}{4N+2}]\Gamma[\frac{1}{2N+1}]}{\sqrt{\pi} (2N+3)\Gamma[\frac{1}{2}+\frac{1}{2N+1}]},$$
(16)

$$b_{2N+3} = -\frac{\hbar}{2},$$
 (17)

$$b_{4N} = \frac{2bi\sin[\frac{\pi}{2N+1}]\Gamma[\frac{2}{2N+1}]}{\sqrt{\pi} (4N+2)\Gamma[\frac{1}{2}+\frac{2}{2N+1}]},$$
(18)

$$b_{4N+6} = \frac{2N\hbar^2 \cos\left[\left(\frac{4N+1}{4N+2}\right)\pi\right]\Gamma\left[1-\frac{1}{2N+1}\right]}{12\sqrt{\pi}\Gamma\left[\frac{1}{2}-\frac{1}{2N+1}\right]}.$$
 (19)

By applying the quantization condition $J(E) = n\hbar$, n = 0, 1, 2, ... the eigenenergies for $V(x) = (ix)^{2N+1} + bx$ can be obtained. Next we demonstrate the accuracy of the AEE method for odd-degree polynomial systems with the help of two Hamiltonians. The first one is the PT symmetric Hamiltonian $H = p^2 + ix^5 + ix$, which possesses real eigen spectrum. Table 1 shows the first 12 eigenvalues of this system obtain with AEE in (15) as well as the numerical eigenenergies obtained with the matrix diagonalization method. It is evident from table 1 that the AEE method for odd-degree polynomial potentials produces accurate real eigenenergies even with four terms. As expected, this method predicts higher eigenenergies more accurately compared to the lower ones.

The second illustration is the non-Hermitian non-PT symmetric system given by $H = p^2 + ix^5 + (1+i)x$.

Table 1. Comparison between calculated energy eigenvalues by AEE and E_{Exact} , which is obtained by the matrix diagonalization method for the Hamiltonian $H = p^2 + ix^5 + ix$ (where $\hbar = 1$).

n	$E_{\rm AEE}$	$E_{\rm Exact}$
0	1.415 221	1.624 377
1	4.868 558	4.820135
2	9.517 600	9.522461
3	15.03904	15.03806
4	21.27666	21.276 58
5	28.133 84	28.13374
6	35.54327	35.543 22
7	43.45471	43.45467
8	51.828 80	51.82877
9	60.633 69	60.633 67
10	69.84293	69.84292
11	79.43411	79.43411

Table 2. Comparison between calculated energy eigenvalues by AEE and E_{Exact} , which is obtained by the matrix diagonalization method for the Hamiltonian $H = p^2 + ix^5 + (1 + i)x$ (where $\hbar = 1$).

n	$E_{ m AEE}$	E_{Exact}
0	1.385 058-0.392 35i	1.529 177-0.552 65i
1	4.857 391-0.499 47i	4.826487-0.45524i
2	9.511 001-0.570 92i	9.514 849-0.573 41i
3	15.034 33-0.625 34i	15.033 80-0.624 25i
4	21.272 98-0.670 02i	21.273 01-0.669 76i
5	28.130 80-0.708 31i	28.13078-0.70813i
6	35.540 68-0.742 04i	35.540 67-0.741 93i
7	43.452 44-0.772 33i	43.452 44-0.772 26i
8	51.82678-0.79992i	51.82677-0.79987i
9	60.631 86-0.825 32i	60.631 86-0.825 28i
10	69.841 26-0.848 90i	69.841 26-0.848 87i
11	79.432.58-0.870.96i	79.432 58-0.870 93i

This system has complex eigen spectrum. Table 2 shows the first 12 eigenvalues of this system obtain with AEE in (15) as well as the numerical eigenenergies obtained with the matrix diagonalization method. It is evident from table 2 that the AEE method for odd-degree polynomial potentials can produce accurate complex eigenenergies even with four terms. Similar to the previous example, this method predicts higher eigenenergies more accurately compared to the lower ones.

The analytic expression of J(E) in (15) can also be utilized to investigate the asymptotic behavior of eigenvalues of potential in (3) analytically.

3. AEE of general odd-degree polynomial potential $V(x) = (ix)^{2N+1} + \beta_1 x^{2N} + \beta_2 x^{2N-1} + \dots + \beta_{2N} x$.

In this section we use the same method used in section 2 to obtain the AEE for the general odd-degree polynomial potential in equation (2). Equation (7) now becomes

$$\frac{\hbar}{i}\frac{\partial P(x,E)}{\partial x} + P^{2}(x,E) = E - (ix)^{2N+1} - \sum_{k=1}^{2N} \beta_{k} x^{2N-k+1}.$$
(20)

Let $\epsilon = E^{-\frac{1}{4N+2}}$ and $y = i\epsilon^2 x$. Then (20) becomes, after simplification,

$$\hbar \epsilon^{4N+4} \frac{\partial P(y,\epsilon)}{\partial y} + \epsilon^{4N+2} P^2(y,\epsilon)$$

= 1 - y^{2N+1} - $\sum_{k=1}^{2N} \beta_k \left(\frac{y}{i}\right)^{2N-k+1} \epsilon^{2k}$. (21)

Now $P(y, \epsilon)$ is expanded as a power series in ϵ ,

$$P(y,\epsilon) = \epsilon^{s} \sum_{n=0}^{\infty} a_{n}(y) \epsilon^{n}, \qquad (22)$$

where s and $a_n(y)$ are determined below. Substituting (22) in (21) and equating coefficients of ϵ^0 , we obtain s = -(2N+1)and $a_0 = \sqrt{1 - y^{2N+1}}$ and (21) becomes

$$\hbar \sum_{n=0}^{\infty} \frac{\mathrm{d}a_n}{\mathrm{d}y} \epsilon^{2N+n+3} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i a_j \epsilon^{i+j}$$
$$= 1 - y^{2N+1} - \sum_{k=1}^{2N} \beta_k \left(\frac{y}{i}\right)^{2N-k+1} \epsilon^{2k}$$
(23)

and rearranging terms, we obtain

$$\left(\hbar \sum_{n=1}^{\infty} \frac{\mathrm{d}a_{n-2N-3}}{\mathrm{d}y} + \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} a_i a_{n-i} + 2a_0 \sum_{n=0}^{\infty} a_n\right)$$

$$\epsilon^n = 1 - y^{2N+1} - \sum_{k=1}^{2N} \beta_k \left(\frac{y}{\mathbf{i}}\right)^{2N-k+1} \epsilon^{2k}.$$
 (24)

Then coefficients a_n are given by

$$a_{n} = \frac{-1}{2a_{0}} \left[\sum_{i=1}^{n-1} a_{i} a_{n-i} + \hbar \frac{\mathrm{d}a_{n-2N-3}}{\mathrm{d}y} + \sum_{k=1}^{2N} \beta_{k} \left(\frac{y}{\mathrm{i}} \right)^{2N-k+1} \delta_{2k,n} \right]. \tag{25}$$

In the above formula $a_n = 0 \forall n < 0$. Now J can be Γ functions as written as

$$J(E) = \sum_{m=0}^{\infty} b_m E^{\frac{-(m-2N-3)}{4N+2}},$$
 (26)

where

$$b_m = \frac{1}{2i\pi} \int\limits_{\gamma} a_m(y) \, \mathrm{d}y. \tag{27}$$

The contour γ encloses the two branch points of $\sqrt{1-y^{2N+1}}$ on the complex plane. The turning points and the contour used are the same as in the previous section. Note that these two branch points lie inside the Stokes wedges which are necessary for defining the above non-Hermitian problem correctly as an eigenvalue problem [7].

We studied explicit expressions for the polynomial potentials of order 3, 5, 7 and 9 and develop a general form for $a_m(y)$. Unlike in the even-degree polynomial potential case, for odd-degrees, there are two forms for the coefficient $a_m(y)$ in (27) depending if *m* is even or odd. Utilizing the procedure described in [18], we have obtain the general expression for $a_m(y)$:

For even m

$$a_{2m}(y) = -y^{\frac{2N-2m+1}{2}} \sum_{j=0}^{2m-1} A_{2m-j-1,j} \left[\frac{y^{(2N+1)/2}}{\left(1 - y^{2N+1}\right)^{1/2}} \right]^{j},$$
(28)

where

$$A_{s,l} = \frac{1}{2} \sum_{t=0}^{l-1} \sum_{i=0}^{s} A_{s-i,l-t-1} A_{i,t} - \frac{1}{4} \hbar i \left(s - 2N \left(l+1\right) - 2\right) \\ \times A_{s-2N-2,l-1} + \frac{1}{4} \hbar i \left(2N+1\right) \left(l-3\right) A_{s-2N,l-3}$$
(29)

and $A_{\alpha+\beta} = 0$ if $\alpha < 0, \beta < 0$ or $\alpha + \beta$ is even. $A_{2N,2} =$ $\frac{(2N+1)i\hbar}{4} \text{ and } A_{4N-2k,1} = \frac{\beta_{2N-k+1}}{2} \text{ for } 0 \leqslant k \leqslant 2N.$ For odd *m*

$$a_{2m+1}(y) = \sum_{j=0}^{2m-1} A_{2m-j-1,j} \left[\frac{y^N}{\left(1 - y^{2N+1}\right)^{1/2}} \right]^j, \quad m \ge 4,$$
(30)

where

$$A_{s,l} = -\frac{1}{4}\hbar i \left(s + l \left(1 - 2N\right) - 2N - 1\right)$$

× $A_{s-2N-1,l-1} + \frac{1}{2}\hbar i N \left(l - 3\right) A_{s-2N+1,l-3}$ (31)

and $A_{\alpha+\beta} = 0$ if $\alpha < 0, \beta < 0$ or $\alpha + \beta$ is even. $A_{2N-1,2} = \frac{iN\hbar}{2}$ and $A_{4N-2k-2,1} = \frac{\beta_{2N-k}}{2}$ for $0 \le k \le 2N - 1$.

Next we obtain the AEE coefficients b_n as

$$b_{2n} = -\frac{1}{2\pi i} \sum_{j=0}^{2n-1} A_{2n-j-1,j} \int_c \frac{y^{\frac{(2N+1)j}{2} + \frac{(2N+1)}{2} - n}}{\left(1 - y^{2N+1}\right)^{j/2}} \,\mathrm{d}y, \qquad (32)$$

$$b_{2n+1}=0.$$

The contour integral in (27) can be evaluated in terms of

$$\int_{c} \frac{y^{\frac{(2N+1)j}{2} + \frac{(2N+1)}{2} - n}}{(1 - y^{2N+1})^{j/2}} dy = \frac{4i^{\frac{2n+j+1}{2}} \cos[(\frac{2Nn+1}{4N+2})\pi]\Gamma[1 - \frac{j}{2}]\Gamma[\frac{3}{2} - \frac{n-1}{2N+1} + \frac{j}{2}]}{(1 + \frac{(2N+1)j}{2} + \frac{(2N+1)}{2} - n)\Gamma[\frac{3}{2} - \frac{n-1}{2N+1}]}.$$
(33)

for odd j and when $\left(\frac{(2N+1)j}{2} + \frac{(2N+1)}{2} - n\right)$ is even. Otherwise, the integral vanishes. Then we have the expression for J(E)as

$$J(E) = -\frac{\hbar}{2} + \sum_{n=0}^{\infty} d_n E^{\frac{-(2n-2N-3)}{4N+2}},$$
 (34)

where

$$d_{n} = -\frac{2\cos\left[\left(\frac{2Nn+1}{4N+2}\right)\pi\right]}{\pi\Gamma\left[\frac{3}{2} - \frac{n-1}{2N+1}\right]} \sum_{j=0}^{2n-1} A_{2(n-j-1),2j+1}$$
$$\times \frac{i^{n+j+1}\Gamma\left[\frac{1}{2} - j\right]\Gamma\left[2 + j - \frac{n-1}{2N+1}\right]}{(1 + (2N+1)(j+1) - n)},$$
(35)

Table 3. Comparison between calculated energy eigenvalues by AEE and E_{Exact} , which are obtained by numerical integration of the Schrödinger equation for the Hamiltonian

 $H = p^2 + (ix)^7 + x^6 + ix^5 + x^2$ (where $\hbar = 1, \beta_1 = 1, \beta_2 = i, \beta_5 = 1$ and $\beta_3 = \beta_4 = \beta_6 = 0$).

п	E_{AEE}	$E_{\rm Exact}$
0	1.738 373 1	1.493 334 2
1	5.174 861 5	5.180 123 0
2	10.480775	10.480 141
3	17.145 262	17.145 492
4	24.970416	24.970 591
5	33.833 532	33.833 555
6	43.647 033	43.647 038
7	54.343 902	54.343 905
8	65.870 552	65.870 553
9	78.182780	78.182781
10	91.243 272	91.243 273
11	105.01995	105.01995

Table 4. Comparison between calculated energy eigenvalues by AEE and E_{Exact} , which is obtained by numerical integration of the Schrödinger equation for the Hamiltonian $H = p^2 + (ix)^9 + x^2 + ix$ (where $\hbar = 1$, $\beta_8 = i$, $\beta_7 = 1$ and $\beta_k = 0$ for all k < 7).

0	·) [·]	
n	$E_{\rm AEE}$	$E_{\rm Exact}$
0	1.845 369 7	1.722 988 2
1	5.730 160 1	5.786 054 6
2	11.834 124	11.847 978
3	19.733 814	19.732 860
4	29.209 838	29.209 369
5	40.121 601	40.121 580
6	52.367 271	52.367 288
7	65.868 043	65.868 047
8	80.560282	80.560 283
9	96.391 051	96.391 052
10	113.315 31	113.315 31

$$d_0 = \frac{2\cos[\frac{\pi}{4N+2}]\Gamma[\frac{1}{2N+1}]}{\sqrt{\pi} (2N+3)\Gamma[\frac{1}{2} + \frac{1}{2N+1}]}$$
(36)

and $A_{2(n-j-1),2j+1}$ is given by (29) and (31).

Note that parameters of the potential are now contained in coefficients $A_{2(n-j-1),2j+1}$ as multinomials in $\beta_1,\beta_2, \ldots, \beta_{2N-1}$.

Now, with the following two examples, the accuracy of the formulas (34)–(36) is tested by calculating eigenvalues of two Hamiltonians using formulas (34)–(36) with the quantization condition $J(E) = n\hbar$, n = 0, 1, 2, ... and comparing them with the exact eigenvalues obtained by numerical integration of the Schrödinger equation.

The first example is the Hamiltonian

$$H = p^{2} + (ix)^{7} + x^{6} + ix^{5} + x^{2}$$
(37)

for this potential N = 3, $\beta_1 = 1$, $\beta_2 = i$, $\beta_5 = 1$ and $\beta_3 = \beta_4 = \beta_6 = 0$. The first 25 non-zero terms of the AEE for this Hamiltonian were obtained from (34)–(36). Table 3 shows the AEE eigenvalues and the exact eigenvalues obtained by numerical integration of the Schrödinger equation for the Hamiltonian (37).

The second example is

$$H = \frac{p^2}{2} + (ix)^9 + x^2 + ix,$$
(38)

where N = 4, $\beta_8 = i$, $\beta_7 = 1$ and $\beta_k = 0$ for all k < 7. The AEE eigenvalues and the exact eigenvalues obtained by numerical integration of the Schrödinger equation for the Hamiltonian (38) are given in table 4.

It is evident from the above two examples that the formulas (34)–(36) derived for odd-degree general polynomial potentials produce very accurate eigenenergies for higher eigenstates. This is due to the fact that the large number of terms in the series could be now included in the calculation with the help of algebraic formulas (34)–(36).

4. Summary and concluding remarks

We derived a simple formula for the semiclassical series for the general polynomial potential $V(x) = (ix)^{2N+1} + \beta_1 x^{2N} + \beta_2 x^{2N}$ $\beta_2 x^{2N-1} + \cdots + \beta_{2N} x$ using the recurrence relations obtained by the AEE method. Almost explicit formulae for the asymptotic eigenenergy expansion are presented for the above potentials for any N. The formulae can be used to find semiclassical analytic expressions for eigenenergies up to any order very efficiently. The formulae are verified with two Hamiltonians. In a previous paper, similar expansions have been obtained for general even-degree polynomial potentials [18]. However, the Hamiltonian for odd-degree polynomial potential considered in this paper is non-Hermitian, and hence the two branch points used for integration must lie within Stokes wedges which are needed for defining the non-Hermitian problems correctly as an eigenvalue problem. It is important to identify that for such systems, the direct application of the WKB method to obtain higher-order terms in the expansion is found to be very complicated (if not impossible) due to the fact that integrals in the expansion coefficients cannot be evaluated analytically. Although the WKB expansion and AEE produce the same semiclassical series for the potential $V(x) = (ix)^{2N+1}$, they are completely different when the potential contains two or more terms. Therefore, the above explicit formula can be employed for obtaining semiclassical eigenspectra in the place of the higher-order WKB method. With the aid of four examples we have shown the accuracy of the AEE method for both real and complex eigen spectra.

The AEE expansions are very useful in analyzing systems analytically. They can be utilized to find out how the level spacings, density of states and other quantities vary with energy and parameters of the potential. Recently the AEE method was found to very valuable in finding isospectral Hermitian and non-Hermitian pairs of Hamiltonians [20]. Since the odd-degree polynomial potentials are non-Hermitian and PT symmetric for certain combinations of parameters $\beta_1, \beta_2, \ldots, \beta_{2N}$ the formulae derived in this paper will be valuable for finding equivalent Hermitian Hamiltonians for non-Hermitian Hamiltonians.

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