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Dynamical tunneling-like effects in 1D classical systems

Asiri Nanayakkara

Institute of Fundamental Studies, Hanthana Road, Kandy, Sri Lanka

E-mail: asiri@ifs.ac.lk

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Abstract

Dynamical tunneling occurs when a particle tunnels between two distinct classically trapped periodic regions of classical phase space that are not separated by a potential barrier. Although the dynamical tunneling has been observed in many multi-dimensional Hamiltonian systems, it has not been observed in 1D systems described by a single potential. In this paper, we show that classical trajectories of real potentials such as $V_1(x) = x^4$ exhibit dynamical tunneling-like behavior when energy or time is complex. It was found that the doubly periodic nature of the Jacobian elliptic functions is responsible for this dynamical tunneling-like behavior. The time spent in one region by the tunneling trajectory before crossing over to the other is found to be proportional to $\left|\frac{E_0^{3/4}}{\Delta E}\right|$, where total energy $E = E_0 + \mathrm{i}\Delta E$ with $E_0 < 0$. Furthermore, we demonstrate that classical trajectories of the non-Hermitian system $V_2(x) = x^4 + (1 + \mathrm{i})x$ show evidence of dynamical tunneling even for real energies. The role of complex time in dynamical tunneling is discussed.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Recently, there has been great interest in classical behavior of 1D Hamiltonian systems in complex phase space and quantum effects in classical systems having complex energies [1–10]. In a recent paper [11], Bender *et al* have shown that some well-known quantum effects can be reproduced qualitatively by means of classical mechanical equations. It was shown that real systems with potential barriers such as the double well $v(x) = x^4 - 5x^2$ can have trajectories that exhibit tunneling-like behavior when energy is taken as a complex quantity. Furthermore, numerically they have found that tunneling-like trajectories of this system first spiral around one pair of turning points and then cross the imaginary axis and spiral around the other pair of turning points. Then, they cross the imaginary axis again and repeat this behavior endlessly.

The quantum tunneling in the above-described double-well system is called barrier tunneling, as tunneling in this system takes place between two regions of a phase which are separated by a barrier in the middle. Many phenomena related to barrier tunneling are widely observed and applied in many areas of physics and chemistry.

It has been found that quantum tunneling can also be observed between two different classically trapped regions of the phase space of polyatomic systems which do not contain potential barriers [12]. When trapping is not due to a potential barrier, the tunneling is dynamical in nature. The dynamical tunneling connects energetically accessible but dynamically separated classical paths in multidimensional systems. This type of tunneling is found to be more subtle than barrier tunneling and the potential function itself will not reveal the clues of classical trapping or the possibility of tunneling.

Although barrier tunneling is not uncommon in one-dimensional systems, and a semiclassical theory of barrier tunneling has been well established [13, 14], dynamical tunneling has not been observed in the real classical phase space of one-dimensional systems described by a single potential. On the other hand, many 1D systems obtained by the reduction of higher dimensional Hamiltonian systems which cannot be globally represented in the form of a particle moving in a potential do exhibit 1D tunneling without a potential barrier.

In this paper, we show that classical trajectories of real potentials such as $V_1(x) = x^4$ exhibit dynamical tunneling-like behavior when energy or time is complex. In section 2, classical trajectories of $V_1(x)$ are investigated. We calculate the time spent in one region by the tunneling trajectory before crossing over to the other. We demonstrate that classical trajectories of the non-Hermitian system $V_2(x) = x^4 + (1+\mathrm{i})x$ show evidence of dynamical tunneling even for real energies in section 3. The role of complex time in dynamical tunneling is discussed in section 4.

2. Classical trajectories of $V_1(x) = x^4$

The classical equation of motion for the potential $V_1(x) = x^4$ is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = \sqrt{E - x^4}, \qquad (1)$$

where p is the classical momentum and E is the total energy. By integrating (1), we have

$$\int \frac{\mathrm{d}x}{\sqrt{E - x^4}} = t + c, \qquad (2)$$

where c is the constant of integration which depends on initial conditions. The left-hand side of the above equation is an elliptic function of the first kind $\mathcal{F}$:

$$\mathcal{F}\left[\sin^{-1}\left(\frac{x(t)}{E^{1/4}}\right); -1\right] = E^{1/4}t + \alpha, \qquad (3)$$

with $\alpha = E^{1/4}c$. We invert the above equation in terms of the Jacobian elliptic function sn [15] as

$$x(t) = E^{1/4} sn(E^{1/4}t + \alpha; -1). \qquad (4)$$

Note that modulus $\kappa^2 = -1$ for the above problem. sn is doubly periodic with periods $4K$ and $2\mathrm{i}K'$, where K and K' are complete elliptic functions defined as

$$K = \int_0^{\pi/2} (1 - \kappa^2 \sin^2(\phi))^{-1/2}\, \mathrm{d}\phi = \int_0^{\pi/2} (1 + \sin^2(\phi))^{-1/2}\, \mathrm{d}\phi, \qquad (5)$$

$$K' \doteq \int_0 (1-t^2)^{-1/2}(1-\kappa'^2 t^2)^{-1/2}\,\mathrm{d}t, \tag{6}$$

and

$$\kappa'^2 = 1-\kappa^2 = 2. \tag{7}$$

Then, K and K' are obtained as

$$K = \frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)} \tag{8}$$

$$K' = \frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}(1-\mathrm{i}). \tag{9}$$

In order to understand the behavior of classical trajectories, we need to relate the periodicity of the Jacobian elliptic function $sn(u)$ to the periodic motion of the trajectory and poles of $sn(u)$ to unbounded motion as follows.

Since Jacobian elliptic function sn has two poles [15] at $\mathrm{i}K'$ and $2K+\mathrm{i}K'$, the trajectory becomes unbounded and the particle escapes to ∞ when one of the conditions $E^{1/4}t+\alpha = \mathrm{i}K'$ and $E^{1/4}t+\alpha = 2K+\mathrm{i}K'$ is satisfied for real t. Therefore, time taken for the particle to escape to ∞ is given by

$$T_\infty = [K(1+\mathrm{i})-\alpha]\,E^{-1/4} = \left[\frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}(1+\mathrm{i})-\alpha\right]E^{-1/4}, \tag{10}$$

$$T'_\infty = [K(3+\mathrm{i})-\alpha]\,E^{-1/4} = \left[\frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}(3+\mathrm{i})-\alpha\right]E^{-1/4}. \tag{11}$$

Since sn is doubly periodic with periods $4K$ and $2\mathrm{i}K'$, periods of the trajectory can be calculated for real E. When $E>0$, the period is given by

$$T = 4KE^{-1/4} = \left[\frac{\sqrt{\pi}\Gamma(1/4)}{\Gamma(3/4)}\right]E^{-1/4}, \tag{12}$$

and for $E<0$

$$T = \frac{4K}{\sqrt{2}}\,|E|^{-1/4} = \left[\frac{\sqrt{\pi}\Gamma(1/4)}{\sqrt{2}\Gamma(3/4)}\right]|E|^{-1/4}. \tag{13}$$

Next we examine the behavior of classical trajectories using above equations. Behavior of the trajectories will depend on the initial conditions. Given an initial condition, first we determine α . If the T_∞ (or T'_∞) is real for this α and period $T > T_\infty$ (or $T > T'_\infty$), then the trajectory will escape to infinity in a finite time T_∞ (or T'_∞). Otherwise, the trajectory is periodic with period T. The relationship between $x(0)$ and α is

$$\alpha = F\left[\sin^{-1}\left(\frac{x(0)}{E^{1/4}}\right); -1\right], \tag{14}$$

or

$$x(0) = E^{1/4} sn(\alpha; -1). \tag{15}$$

First, we consider the case when $E>0$. Let $\alpha = \alpha_r + \mathrm{i}\alpha_i$. T_∞ and T'_∞ in (10) and (11) are real only if $\alpha_i = K$. That is, $\alpha = \alpha_r + \mathrm{i}K$. Now we rewrite $sn(\alpha_r+\mathrm{i}K;-1)$ in terms of Jacobian elliptic functions $nd(\alpha_{r;}; \kappa')$:

$$sn(\alpha_r+\mathrm{i}K;-1) = \mathrm{i}\, nd(\alpha_r; 2). \tag{16}$$

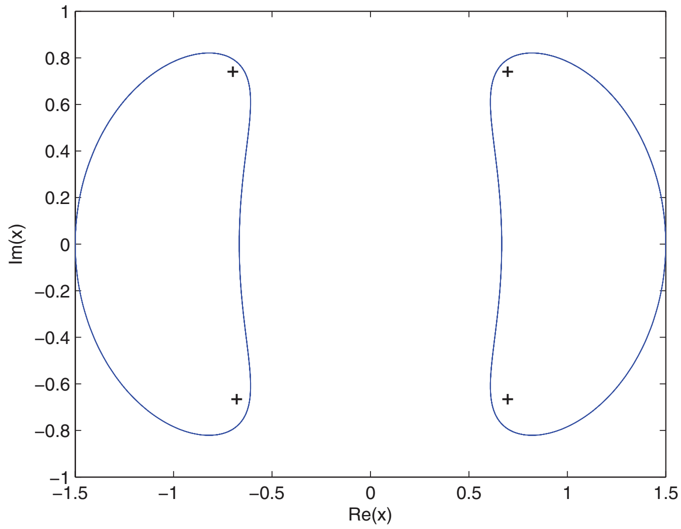


Figure 1. Two classical trajectories of the potential $V(x) = x^4$ in the complex x plane corresponding to the energy $E = -1$. Both trajectories are periodic and trapped in a respective pair of turning points.

Note that here we have used the properties $sn(\alpha_r + \mathrm{i}K; -1) = \mathrm{i}\, sn(\mathrm{i}\alpha_r + K; -1)$ and $sn(\mathrm{i}\alpha_r + K; -1) = cd(\mathrm{i}\alpha_r)$. Since $nd(\alpha_r; 2)$ is real for real α_r, $sn(\alpha_r + \mathrm{i}K; -1)$ must be pure imaginary. Furthermore, $|nd(\alpha_r; 2)| > 1$ for all real α_r and hence $|sn(\alpha_r + \mathrm{i}K; -1)| > 1$. Consequently, in order for T_∞ be real, $x(0)$ has to be pure imaginary and $|x(0)| > E^{1/4}$. This implies that when $E > 0$, all the trajectories starting from the imaginary x axis above $E^{1/4}$ and below $-E^{1/4}$ will escape to infinity in time T_∞ and all the other trajectories are periodic.

Next we assume $E < 0$. T_∞ now becomes

$$T_\infty = [2K - \alpha(1 - \mathrm{i})]|E|^{-1/4}. \tag{17}$$

Let $\alpha = \alpha_r + \mathrm{i}\alpha_i$:

$$T_\infty = [2K - (\alpha_r + \alpha_i) + \mathrm{i}(\alpha_i - \alpha_r)]|E|^{-1/4}. \tag{18}$$

For $E < 0$,

$$x(0) = |E|^{1/4}(1 + \mathrm{i})sn(\alpha; -1)/\sqrt{2}. \tag{19}$$

Therefore, T_∞ is real when $\alpha_r = \alpha_i$. On the other hand, $sn(a(1 + \mathrm{i}); -1)$ (a is real) can be written as $b(1 + \mathrm{i})sn(a; -1)$ (for some $b \in \mathbb{R}$). As a result, we have

$$x(0) = \sqrt{2}\mathrm{i}|E|^{1/4}sn(a; -1). \tag{20}$$

Therefore, for $E < 0$ any trajectory starting from the imaginary x axis will escape to infinity in time T_∞ and all the other trajectories are periodic. Two typical periodic trajectories when $E < 0$ are shown in figure 1. Each one of the two trajectories is trapped in a respective pair of turning points as shown.

Next we introduce a small complex component to the total energy E_0. First consider the case $E_0 < 0$. Then, total energy becomes $E = -E_0 + \mathrm{i}\Delta E$ for some real ΔE:

$$x(t) = (-E_0 + \mathrm{i}\Delta E)^{1/4}sn((-E_0 + \mathrm{i}\Delta E)^{1/4}t + \alpha; -1). \tag{21}$$

Due to the complex component $\mathrm{i}\Delta E$, trajectories which are periodic for real energies have now become non-periodic. Assuming ΔE is small compared to E_0, we expand $(-E_0+\mathrm{i}\Delta E)^{1/4}$ as

$$(-E_0+\mathrm{i}\Delta E)^{1/4} = \frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i}) + \frac{\Delta E}{4\sqrt{2}E_0^{3/4}}(1-\mathrm{i}). \tag{22}$$

Let $\Delta\epsilon = \frac{\Delta E}{4\sqrt{2}E_0^{3/4}}(1-\mathrm{i})$, and $x(t)$ becomes

$$x(t) = A\, sn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t + \Delta\epsilon t + \alpha; -1\right), \tag{23}$$

where $A = \left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})+\Delta\epsilon\right)$. First consider a trajectory starting from the origin. Then, $\alpha = 0$. The case when $\alpha \neq 0$ will be considered later. Next we monitor $x(t)$ as time t increases from zero. We approximate $x(t)$ as

$$x(t) = A\, sn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t; -1\right) + A\Delta\epsilon t\, cn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t; -1\right) dn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t; -1\right) \tag{24}$$

Here, we have used the relation $\frac{\mathrm{d}(sn(u))}{\mathrm{d}u} = cn(u)\, dn(u)$, where $cn(u)$ and $dn(u)$ are Jacobian elliptic functions.

Since $sn(u+2m\mathrm{i}K') = sn(u)$, $cn(u+2m\mathrm{i}K') = (-1)^m cn(u)$ and $dn(u) = (-1)^m\, dn(u+2m\mathrm{i}K')$ for integer m, when $t = \frac{2\sqrt{2}mK}{E_0^{1/4}}$, $sn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t; -1\right)$ becomes zero.

Since $2m\mathrm{i}K' = 2mK(1+\mathrm{i})$, $sn(0) = 0$, $cn(0) = 1$ and $dn(0) = 1$, then $x(t)$ becomes

$$x(t) \approx A\Delta\epsilon t \neq x(0). \tag{25}$$

Therefore, the trajectory is not periodic but it spirals. As t increases further, the quantity $\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t + \Delta\epsilon t$ will become large enough to satisfy the condition

$$\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t + \Delta\epsilon t = 2nK + 2m\mathrm{i}K' = 2nK + 2mK(1+\mathrm{i}), \tag{26}$$

where n and m are non-negative integers. Note that condition (26) cannot be satisfied by any real t when $m = 0$ as $|\Delta\epsilon|$ is small. $sn(u+2nK+2m\mathrm{i}K') = -sn(u)$ and hence for any τ which satisfies (26)

$$x(\tau) \approx (-1)^n x(0). \tag{27}$$

Since n represents the number of crossings by imposing the condition that τ must be real and $n = 1$, we find m, which is the approximate number of times the trajectory spirals before it crosses from one pair of turning points to the other.

Figure 2 shows a typical dynamical tunneling trajectory for the system. Equation (26) can be written in terms of ΔE as

$$\mathrm{Re}(\tau) = \frac{1}{\sqrt{2}E_0^{1/4}}\left\{2K + 4mK + \frac{\Delta E}{2E_0}K\right\}, \tag{28}$$

and

$$\mathrm{Im}(\tau) = \frac{1}{\sqrt{2}E_0^{1/4}}\left\{\frac{\Delta E}{2E_0}(2m+1)K - 2K\right\}. \tag{29}$$

By imposing the condition that $\mathrm{Im}(\tau) = 0$, we have

$$m = \frac{2E_0}{\Delta E} - \frac{1}{2}, \tag{30}$$

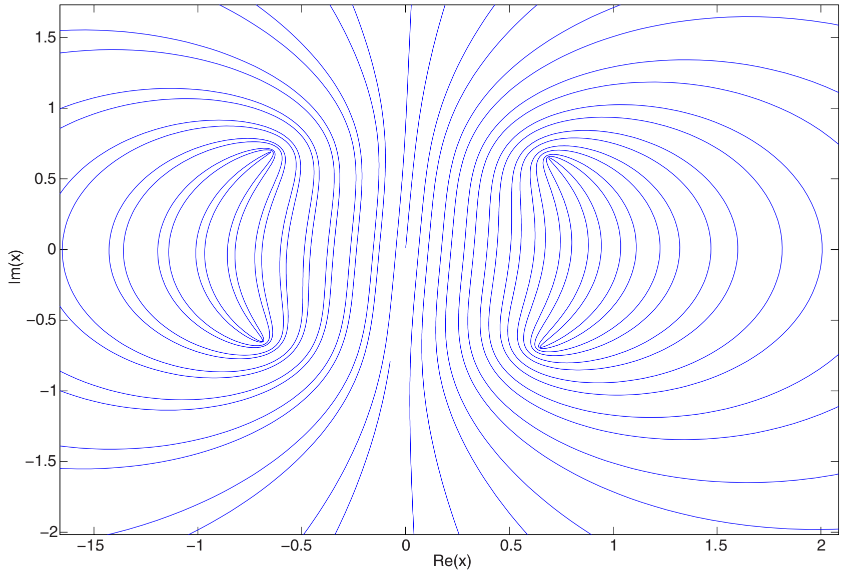


Figure 2. A typical tunneling-like trajectory of $V(x) = x^4$ corresponding to $E = -0.8 + 0.1\mathrm{i}$. This trajectory starts from the origin. First it spirals around the right pair of turning points for a while, and then it moves across the imaginary axis and spirals around the left pair of turning points. After spending some time, it goes across the imaginary axis again to the right-hand side and spirals again around the right pair turning points, repeating the same process endlessly.

and hence the time spent by the trajectory before crossing to the other side is given by

$$T_c \approx \frac{\sqrt{2\pi}\,\Gamma(1/4)}{\Gamma(3/4)}\left[\frac{E_0^{3/4}}{\Delta E}\right]. \tag{31}$$

We have derived the above results for $\alpha = 0$. When $\alpha \neq 0$ equations (25) and (26) become

$$x(t) \approx A\Delta\epsilon cn(\alpha)\,dn(\alpha)t \neq x(0), \tag{32}$$

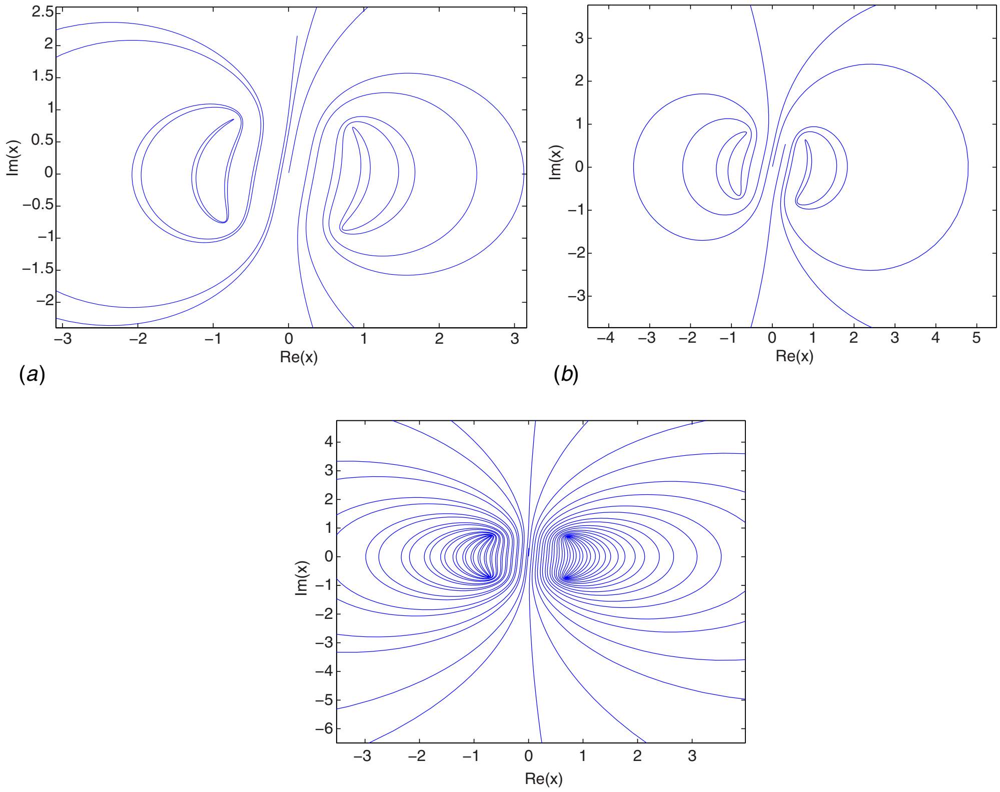
$$\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t + \Delta\epsilon t = 2nK + 2mK(1+\mathrm{i}) - \alpha. \tag{33}$$

Using equations (28)–(33), we make the following observations.

(a) Any trajectory starting from the origin will spiral around one pair of turning points for $\frac{\sqrt{2\pi}\Gamma(1/4)}{\Gamma(3/4)}\left[\frac{E_0^{3/4}}{\Delta E}\right]$ amount of time before crossing over to the other pair of turning points. Since the real part of the energy is negative, when the real part of the total energy decreases (i.e. increase of E_0) the trajectory will cross over less frequently. Similarly, the smaller the complex energy ΔE, the longer the time the trajectory will spiral around one pair of turning points. This is illustrated in figure 3. Furthermore, when $\Delta E \to 0$, $\frac{\sqrt{2\pi}\Gamma(1/4)}{\Gamma(3/4)}\left[\frac{E_0^{3/4}}{\Delta E}\right] \to \infty$, and the particle will be trapped around one pair of turning points forever and no tunneling will take place.

(b) If the trajectory does not start from the origin, then the amount of time it will spend around the *first* pair of turning points will depend on the starting location of the trajectory. However, after the first crossing, it will spend the same amount of time around each pair of turning points as in (*a*) (i.e. $T_c = \frac{\sqrt{2\pi}\Gamma(1/4)}{\Gamma(3/4)}\left[\frac{E_0^{3/4}}{\Delta E}\right]$).

Using techniques similar to those described before, it can be shown that if the real part of the energy is positive, then no tunneling-like behavior can be observed.



(*a*) (*b*)

(*c*)

Figure 3. Behavior of tunneling-like trajectories as energies E_0 and ΔE change. The trajectory in (*a*) represents $E = -1.5 + 0.5\mathrm{i}$. The trajectory in (*b*), which corresponds to higher real energy $E = -1.0 + 0.5\mathrm{i}$, spends less time spiraling around the right pair turning points before crossing over to the other side. On the other hand, when the imaginary part of the energy ΔE was decreased while keeping the real part of the energy fixed at $-1(E = -1.0 + 0.1\mathrm{i})$, the time spent by the trajectory becomes large as shown in (*c*).

3. Classical trajectories of $V_2(x) = x^4 + (1 + \mathrm{i})x$

The classical equation of motion for the potential $V_2(x) = x^4 + (1 + \mathrm{i})x$ is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = \sqrt{E - x^4 - (1 + \mathrm{i})x}. \tag{34}$$

In order to investigate tunneling-like behavior in this system, we solve (34) numerically. Although it can be solved analytically in terms of Jacobian elliptic functions as in the case of $V_1(x)$, we used the numerically generated trajectories to understand the behavior qualitatively. For real positive energy no tunneling-like behavior was observed. Figure 4 shows trajectories for three different real negative energies. Dynamical tunneling-like trajectories can be observed in this non-Hermitian system even for negative real energies without an imaginary part as shown in figure 4. As in the case of x^4 potential, for lower energies, trajectories get trapped in one pair of turning points for a longer time period before tunneling compared to the higher energies.

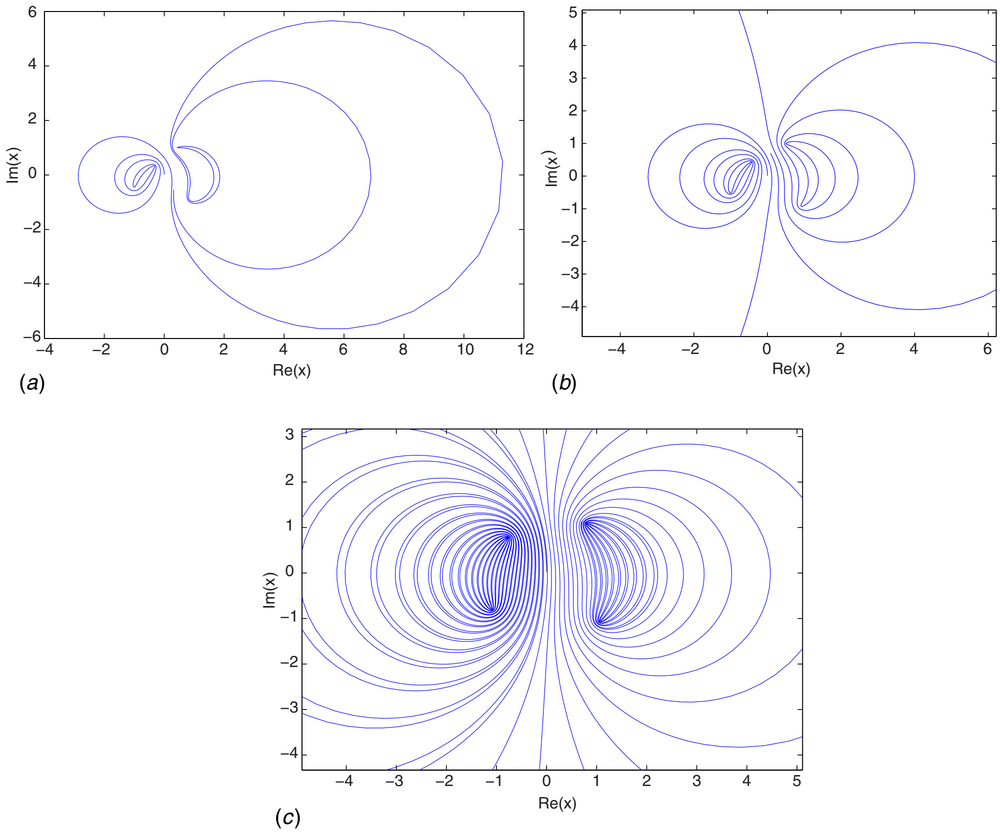


Figure 4. Tunneling-like trajectories of $V(x) = x^4 + (1+\mathrm{i})x$ for real negative energies. Energies of trajectories corresponding to (*a*), (*b*) and (*c*) are $E = -0.7$, $E = -1.0$ and $E = -3.0$, respectively. These trajectories clearly demonstrate that as energy decreases, the amount of time spent by each trajectory around one pair of turning points will increase similar to the case of barrier tunneling.

4. Dynamical tunneling and complex time

When energy is complex, tunneling-like behavior in classical trajectories for barrier potentials such as $V(x) = x^4 - bx^2$ has been previously observed [11, 16]. In this section, we examine dynamical tunneling-like behavior of classical trajectories for the x^4 potential and $x^4 + bx$ (where b is real) when time is complex. For x^4 potential, classical trajectories are described by

$$x(t) = E_0^{1/4} sn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t + \alpha; -1\right), \tag{35}$$

when energy is real and negative. Here, we rewrite equation (23) without the $\Delta\epsilon$ term. Next we replace t by $t_0\,\mathrm{e}^{-\mathrm{i}\Delta\theta}$, where $\Delta\theta$ is small. It is easy to see that now equation (35) becomes similar to equation (23):

$$x(t_0\,\mathrm{e}^{\mathrm{i}\Delta\theta}) = E_0^{1/4} sn\left(\frac{E_0^{1/4}}{\sqrt{2}}(1+\mathrm{i})t_0 + \Delta\epsilon t_0 + \alpha; -1\right), \tag{36}$$

but $\Delta\epsilon$ is now given by

$$\Delta\epsilon = \frac{E_0^{1/4}}{\sqrt{2}}(1-\mathrm{i})\Delta\theta. \tag{37}$$

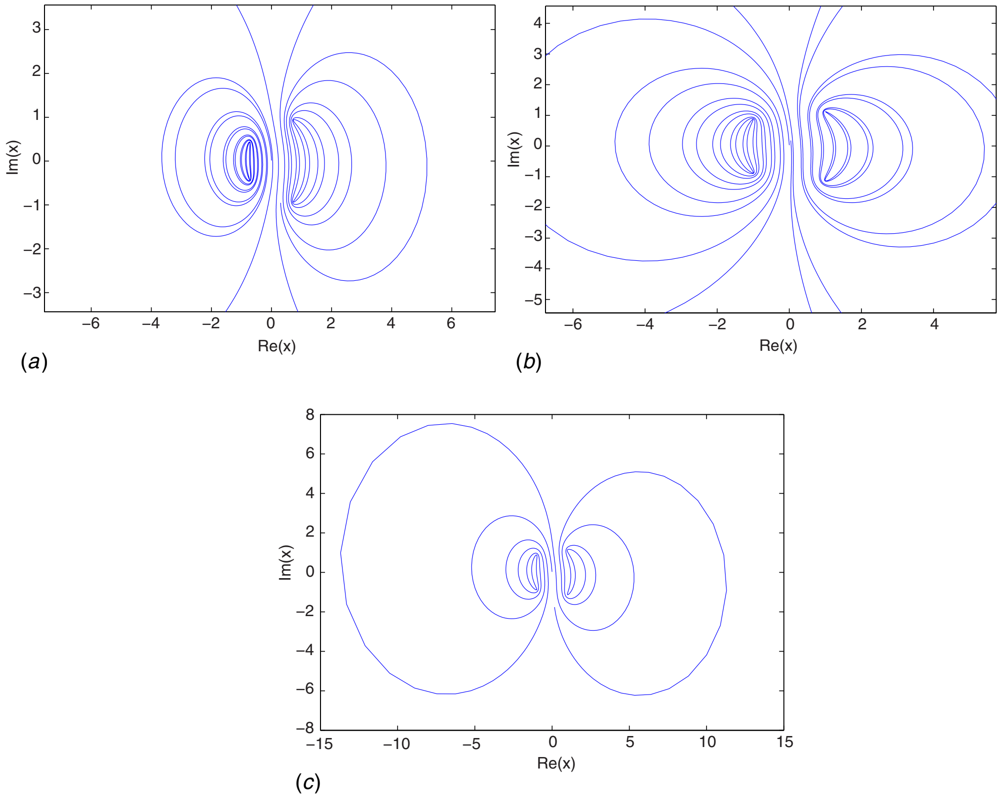


Figure 5. Classical tunneling-like trajectories of the real Hermitian potential $V(x) = x^4 + bx$ for real b and complex time $t = t_0\,\mathrm{e}^{\mathrm{i}\Delta\theta}$. Real energies and complex phase angles $\Delta\theta$ corresponding to (*a*), (*b*) and (*c*) are ($E = -1.0$ and $\Delta\theta = 0.05$), ($E = -4.0$ and $\Delta\theta = 0.05$) and ($E = -4.0$ and $\Delta\theta = 0.1$), respectively. It is evident from these trajectories that the amount of time spent by trajectories around one pair of turning points is less sensitive to change in energy as compared to the changes in the complex phase angle.

Here we have used the approximation

$$t = t_0\,\mathrm{e}^{-\mathrm{i}\Delta\theta} \approx t_0(1 - \mathrm{i}\Delta\theta). \tag{38}$$

Now m and T_c as in equations (30) and (31) become

$$m = \frac{1}{2\Delta\theta} - \frac{1}{2}, \tag{39}$$

$$T_c \approx \frac{\sqrt{2\pi}\,\Gamma(1/4)}{4\Gamma(3/4)E_0^{1/4}}\left[\frac{1}{\Delta\theta}\right]. \tag{40}$$

It is evident from equation (36) that when time is complex and energy is real, certain trajectories show dynamical tunneling-like behavior. However, m is now independent of the energy E_0 as shown in (39). In other words, for complex time, the number of times the trajectory spirals before it crosses from one pair of turning points to the other is independent of energy and depends only on the phase angle of the time t.

Next consider the potential $x^4 + bx$ (where b is real). The classical equation of motion for this system is solved by numerical integration. Three tunneling-like trajectories are shown in figures 5(*a*)–(*c*). In all three cases, energy is real and negative, and the phase angle introduced for time is small. These figures show that the quantities m and T_c change very little as energy changes by a large amount. On the other hand, when $\Delta\theta$ changes, m and T_c change considerably.

5. Summary and concluding remarks

In this paper, we have shown that quantum dynamical tunneling-like behavior, which takes place in multidimensional real Hermitian quantum systems, can occur in 1D systems when energy or time is complex. For the barrierless potential $V(x) = x^4$, we obtained an analytic expression for the motion of the particle. The number of times a classical trajectory spirals around before it crosses from one pair of turning points to the other is $\left[\frac{2E_0}{\Delta E} - \frac{1}{2}\right]$, and the amount of time spent by the trajectory before crossing over to the other side is found to be $\frac{\sqrt{2\pi}\Gamma(1/4)}{\Gamma(3/4)}\left[\frac{E_0^{3/4}}{\Delta E}\right]$, where $-E_0$ is the real part of the total energy and ΔE is the imaginary part of the total energy. The dynamical tunneling-like behavior in this system is a result of the doubly periodic nature of the Jacobian elliptic functions.

The dynamical tunneling-like behavior was also observed in the non-Hermitian systems such as $V(x) = x^4 + (1+\mathrm{i})x$ even for real negative energies. As in the case of barrier tunneling, for low energies, particle trajectories get trapped in one pair of turning points for a long time before tunneling over to the other pair of turning points. It was found that real Hermitian potentials such as $V(x) = x^4 + bx$, where $b \in \mathbb{R}$, can have classical trajectories which tunnel between a pair of turning points for real energies when time contains a small imaginary part. In a recent paper [11], Bender *et al* suggested a possibility of having both real and imaginary parts for ΔE in the uncertainty relation $\Delta E \Delta t \geqslant \hbar/2$. In this paper and [17], we see the similarity between the behavior of classical trajectories when energy and time are complex. Therefore, we believe that there is also a possibility of having both real and imaginary parts for Δt in the uncertainty relation above.

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