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Maneuvering periods of 2D quantum walks with the coin operator

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Abstract. Recurrence in classical random walks is well known and the idea has been investigated in quantum walks in many aspects. The recurrence in quantum walks is termed when the walker returns to the origin with a nonzero probability and if the final coin state is also the same as the initial coin state then the quantum walk is said to have a full revival. So far, full revival 2D quantum walks with a period larger than two steps have not been found and it has been argued that four-state quantum walks cannot have periods longer than two steps. In this paper, with the aid of simple 2D non-local coins we show that some four-state quantum walks can have full revivals with any even period and the periodicity can be controlled with a slight change of a single parameter within the coin operator.

1 Introduction

Quantum walks (QWs) are the quantum equivalent of Classical Random Walks (CRW) which provide useful tools for powerful algorithm developments and testing grounds for various physical phenomena. Diverse types of quantum walks and their properties such as recurrence of states, localization, entanglement, walks on Graphs and cycles have been investigated [1–12] and used as resources in quantum computing. Quantum walks on cycles can be used as simple powerful models for investigating quantum as well as classical-quantum hybrid networks [13]. In coin based quantum walks among many coin operators, Hadamard, Grover, and Fourier Coins are the widely used coin operators. The distributions of quantum walks due to these coins have been rigorously studied [1–12]. The Grover walk is noticeable as the walk generated by Grover coin exhibits localization regardless of whether the initial state is a product state or an entangled Bell state. In addition, recently, a new set of non-local coins has been introduced to study the manifestation of classical conditions imposed upon CRWs in corresponding QWs [14].

In both CRWs and QWs, one of the interesting properties is the returning of the walker to the original location. The recurrence in the quantum walk takes place when the probability of returning the walker to the origin is nonzero. However the final state of the coin does not have to be the same as the initial state. If the final coin state is the same as the initial state of the coin then the walk is said to have full revivals. This recurrence in classical random walks is characterized by the Polya number and it is generalized for quantum walks by Štefaňák *et al.* in [15].

Dimensionality of the lattice, the choice of the coin operator as well as the initial coin state of the walker determine the recurrence of quantum walks in [15]. The Grover walk is generalized in [16] to show that one can construct a quantum walk in arbitrary dimensions which is recurrent even though the classical walks are recurrent only for the dimensions $d = 1, 2$. Štefaňák *et al.* in [17] have determined the range of parameters for which the biased quantum walks remain recurrent and found that there exit genuine biased quantum walks that are recurrent. The walk having unequal step lengths is defined as the *genuine biased* quantum walk [17].

So far, full revival 2D quantum walks with a period larger than two steps have not been found. Further Štefaňák *et al.* have argued in [18] that revivals with a longer period than two steps cannot be achieved for a four-state quantum walk. In this paper, with the aid of simple 2D non-local coins we show that some four-state quantum walks can have full revivals with any even periods.

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2 2D quantum walks with non-local coins

Let us consider a two-particle system. The Hilbert space of the two particles can be expressed as

$$H = H_p \otimes H_c. \tag{1}$$

The position Hilbert space H_p is spanned by the vector $|x, y\rangle$ where x and y represent the location of the two particles. The coin space H_c is spanned by $|i, j\rangle$ where i and j represent the coin states $(\{|i\rangle; 0, 1\})$. Then the wave function of the two particles can be expressed as

$$|\psi(t)\rangle = \sum_{x,y} \sum_{i,j} a_{x,y;i,j}(t)|x, y; i, j\rangle. \tag{2}$$

The time evolution operator is given by

$$U = S(I \otimes C), \tag{3}$$

where C denotes the coin operator and the shift operator S is given by

$$S = \sum_{x,y=-\infty}^{\infty} (|x+1, y+1\rangle\langle x, y| \otimes |0, 0\rangle\langle 0, 0| + |x+1, y-1\rangle\langle x, y| \otimes |0, 1\rangle\langle 0, 1| \\ + |x-1, y+1\rangle\langle x, y| \otimes |1, 0\rangle\langle 1, 0| + |x-1, y-1\rangle\langle x, y| \otimes |1, 1\rangle\langle 1, 1|). \tag{4}$$

In this paper we focus on non-local coins. In [14] a class of non-local quantum coins which reflect classical conditions has been introduced as

$$C_N(\alpha, \theta, \phi) = A_1 \otimes B_1 + A_2 \otimes B_2, \tag{5}$$

where $A_1 = \left[\begin{smallmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{smallmatrix}\right]$, $B_1 = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right]$, $B_2 = \left[\begin{smallmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{smallmatrix}\right]$, and $\alpha, \theta, \phi \in [0, 2\pi]$. Note that the coin $C_N(\alpha, \theta, \phi)$ is unitary $\forall \alpha, \theta, \phi \in [0, 2\pi]$ while A_1, A_2 are non-unitary and B_1, B_2 are unitary.

The four-state Grover coin is also a non-local coin. It cannot be written as a tensor product of two one-dimensional coins but can be written in the form given in (5) as a summation of two tensor products, each term containing a unitary and a non-unitary coin as $G_{2D} = I \otimes E + G_{1D} \otimes F$, where $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $E = \frac{1}{2}\left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, $F = \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $G_{1D} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

In this paper, we study QWs governed by the coins $C_N(\frac{\pi}{2}, \theta, \frac{\pi}{2})$, $\theta \in [0, 2\pi]$ and compare the results with QWs based on the Grover coin G

$$G = \begin{pmatrix} E & F \\ F & E \end{pmatrix}. \tag{6}$$

The coin $C_N(\frac{\pi}{2}, \theta, \frac{\pi}{2})$ can be expressed in the terms G_{1D} and B_1 as

$$C_N\left(\frac{\pi}{2}, \theta, \frac{\pi}{2}\right) = \begin{pmatrix} O & G_{1D} \\ B_1 & O \end{pmatrix}, \tag{7}$$

where O is the zero matrix.

As mentioned earlier the coin $C_N(\frac{\pi}{2}, \theta, \frac{\pi}{2})$ is unitary. The state of the two particles after time t is obtained by applying the evolution operator U on the initial state of the walker.

3 Conditions for full revivals

First, Fourier transform is carried out on the evolution operator and obtain the k, l momentum representation of U. The time evolution in the Fourier picture simplifies to

$$U_{k,l} = D(k, l) \cdot C, \tag{8}$$

$$D(k, l) = D(k) \otimes D(l), \tag{9}$$

where $D(\cdot) = \mathrm{Diag}(e^{i(\cdot)}, e^{-i(\cdot)})$

Periodicity is understood as the full revival of a quantum state. A quantum walk is evolved under a evolution operator acting on the state of the particle. The final state $|\psi(T)\rangle$ of the walkers after T steps in terms of the initial state $|\psi(0)\rangle$ is

$$|\psi(T)\rangle = U_{k,l}^T |\psi(0)\rangle. \tag{10}$$

Table 1. The first 6 steps of the QW generated by the coin $C_N(\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{2})$ for a product state ($\psi_i = (0,1,0,0)^T \otimes |0;0\rangle$) at the origin as the initial state. $|\psi(t)\rangle = (a_{00},a_{01},a_{10},a_{11})^T \otimes |x;y\rangle$ represents the respective coin states in the location (x,y) at time t.

Quantum state
$\|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\otimes\|x=0;y=0\rangle$
$\|\psi(t=1)\rangle = \frac{1}{\sqrt{8}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\otimes\|x=-1;y=-1\rangle - \sqrt{\frac{3}{8}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\otimes\|x=-1;y=1\rangle - \frac{1}{\sqrt{8}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\otimes\|x=1;y=1\rangle$
$\|\psi(t=2)\rangle = -\frac{1}{\sqrt{8}}\begin{pmatrix}\sqrt{3}\\-1\\1\\0\end{pmatrix}\otimes\|x=0;y=0\rangle + \sqrt{\frac{3}{8}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\otimes\|x=0;y=2\rangle$
$\|\psi(t=3)\rangle = -\frac{1}{\sqrt{8}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\otimes\|x=-1;y=-1\rangle - \sqrt{\frac{3}{8}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\otimes\|x=-1;y=1\rangle + \frac{1}{\sqrt{8}}\begin{pmatrix}\sqrt{3}\\-1\\0\\0\end{pmatrix}\otimes\|x=1;y=1\rangle$
$\|\psi(t=4)\rangle = -\frac{1}{\sqrt{8}}\begin{pmatrix}\sqrt{3}\\1\\-1\\0\end{pmatrix}\otimes\|x=0;y=0\rangle + \sqrt{\frac{3}{8}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\otimes\|x=0;y=2\rangle$
$\|\psi(t=5)\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\otimes\|x=-1;y=-1\rangle + \frac{1}{\sqrt{8}}\begin{pmatrix}\sqrt{3}\\1\\0\\0\end{pmatrix}\otimes\|x=1;y=1\rangle$
$\|\psi(t=6)\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\otimes\|x=0;y=0\rangle$

Note that for an arbitrary initial state $|\psi(0)\rangle$, $|\psi(T)\rangle = U_{k,l}^T|\psi(0)\rangle = |\psi(0)\rangle$ if and only if each of the eigenvalues (λ_i) of $U_{k,l}$ simultaneously satisfies

$$\lambda_i^T = 1, \tag{11}$$

leading to

$$U_{k,l}^T = I, \tag{12}$$

for some time T.

By imposing the condition given in (12) on the coin $C_N(\frac{\pi}{2},\theta,\frac{\pi}{2})$ we obtain two conditions on θ for the quantum walk to become periodic. When θ has the form $\theta = \frac{n}{2m}\pi$, n, $m \in \mathbb{Z}$ and $0 < n \le 4m$, then the period is $4m$. On the other hand, when $\theta = \frac{(4n\pm1)}{2m+1}\pi$, $n, m \in \mathbb{Z}$ and $0 < n \le 2m$, the period is $4m+2$. Note that any even period can be obtained by choosing suitable n and m with the above formulae and hence there is an angle θ of the coin operator corresponding to every even period. Therefore there can be periods longer than 2 for four-state quantum coins and as $m \to \infty$ there are infinitely many even periods. It is interesting to note that the periodicity is very sensitive to the angle parameter θ in the coin operator and the period can be controlled by a large amount with a slight change of θ. As an example consider the case when $n=3$ and $m=5$ with the formula $\theta = \frac{n}{2m}\pi$. For this combination $\theta = 0.3\pi$ and period is $4m = 20$. If we reduce the angle θ by an amount 0.0001π, the new angle $\widetilde{\theta} = 0.2999\pi = 2999\pi/10000$ and hence new $n = 2999$ and $m = 5000$ giving the period 20000. Furthermore, there are infinitely many angles θ (*e.g.*, $\theta = a\pi$ where a is irrational) for which the quantum walk is non-periodic (having infinite periods). Since rational numbers are dense in real numbers, a rational number r can be found arbitrary close to any irrational a (*i.e.*, $|r-a| < \epsilon$ for arbitrary $\epsilon > 0$) and hence one can generate arbitrarily large periods.

In this quantum walk, if the initial state is a product state, during the subsequent walk the walker can be found only at two possible locations in the x-direction and three possible locations in the y-direction. On the other hand, if it is a Bell state the walker can have only three possible locations in the x-direction and four possible locations in the y-direction before repeating the pattern as evident from tables 1 and 2. $|\psi(t)\rangle = (a_{00},a_{01},a_{10},a_{11})^T \otimes |x;y\rangle$ represents the respective coin states in the location (x,y) at time t.

Table 2. The first 6 steps of the QW generated by the coin $C_N(\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{2})$ for a Bell state ($\psi_i = \frac{1}{\sqrt{2}}(0,1,-1,0)^T \otimes |0;0\rangle$) at the origin as the initial state. $|\psi(t)\rangle = (a_{00},a_{01},a_{10},a_{11})^T \otimes |x;y\rangle$ represents the respective coin states in the location (x,y) at time t.

Quantum state
$\lvert\psi(t=0)\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \lvert x=0;y=0\rangle$
$\lvert\psi(t=1)\rangle = \frac{1}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \lvert x=-1;y=-1\rangle - \frac{\sqrt{3}}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \lvert x=-1;y=1\rangle$
$\lvert\psi(t=2)\rangle = \frac{1}{2}\begin{pmatrix}-\sqrt{3}\\1\\0\\0\end{pmatrix} \otimes \lvert x=0;y=0\rangle$
$\lvert\psi(t=3)\rangle = -\frac{1}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \lvert x=-1;y=-1\rangle - \frac{\sqrt{3}}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \lvert x=-1;y=1\rangle$
$\lvert\psi(t=4)\rangle = -\frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\\0\\0\end{pmatrix} \otimes \lvert x=0;y=0\rangle$
$\lvert\psi(t=5)\rangle = -\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \lvert x=-1;y=-1\rangle$
$\lvert\psi(t=6)\rangle = -\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \lvert x=0;y=0\rangle$

According to tables 1 and 2, when $\theta = \frac{\pi}{6}$ the quantum walk has a period of 12 for both Bell and product initial states. In this example, $|\Psi(t=6)\rangle = -|\Psi(t=0)\rangle$ and the steps from $t=0$ to $t=6$ repeat until $t=12$ and $|\Psi(t=12)\rangle = |\Psi(t=0)\rangle$. In other words the walker returns to its exact initial state after 12 time steps. Note that the Grover coin also shows localization effects for all initial states except $\psi_i = \frac{1}{2}(1,-1,-1,1)^T$ with period 2 whereas the above walk with $C_N(\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{2})$ shows full revivals independent of the initial state with period 12.

Štefaňák *et al.* in [18] have proved that there cannot be full revivals of a quantum state with periods longer than two steps for a four-state quantum coin. In their proof they have not considered the effects of the coin parameters on the characteristic polynomial of $U_{k,l}$ for the most general case. Hence they could only obtain two different eigenvalues. The coin $C_N(\frac{\pi}{2},\theta,\frac{\pi}{2})$ is an example for a four-state quantum coin which produces full revivals longer than two steps as it gives four different eigenvalues for the characteristic polynomial of $U_{k,l}$. It is worth noting that all four eigenvalues of the propagator of the quantum walk which is governed by the coin $C_N(\frac{\pi}{2},\theta,\frac{\pi}{2})$ are constant and independent of momenta k and l. In [18], this case was not considered seriously and discarded. Another difference between the present work and ref. [18] is in their definition of the shift operator: while the present work assumes tensor product structure $D(k,l) = D(k) \otimes D(l)$ (in the momentum representation), ref. [18] uses a direct sum $D(k,l) = \mathrm{Diag}(e^{ik}, e^{-ik}, e^{il}, e^{-il})$. However their final result was not affected by this (*i.e.*, for their example both shift operators produce the same result).

4 Conclusion

The full revival of a quantum walk emphasizes the difference between classical and quantum walks. Here we have been able to govern the revival of the quantum walk by a simple change in the coin operator. The parameter θ of the coin gives the freedom to manipulate the periodicity of the walk as desired. Until now full revival of a four-state quantum walk with a period larger than two steps has not been found. It has been claimed in [18] that revivals with a longer period than two steps cannot be achieved for a four-state quantum walk. In this paper we showed that quantum walks governed by the coin $C_N(\frac{\pi}{2},\theta,\frac{\pi}{2})$ can produce full revivals with any even periods thereby giving a counter example for the above-mentioned claim. Our curiosity to produce full revivals with periods longer than two steps has been purely academic and hope that these results will help to build new quantum algorithms.

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