Stability of the Biomass–Carbon Dioxide Equilibrium in the Atmosphere: Mathematical Model

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ABSTRACT

A simple mathematical model is constructed to determine the stability conditions of the biomass–carbon dioxide equilibrium of the atmosphere. The model predicts that carbon dioxide emissions from fossil fuels in general will not disturb the equilibrium. However, severe deforestation could lead to an instability with a rapid increase in the carbon dioxide concentration.

INTRODUCTION

The carbon dioxide pollution of the atmosphere is a problem of tremendous importance that has attracted the attention of international research organizations and workers in different disciplines [1–6]. Burning of fossil fuels and deforestation [7] are believed to increase the carbon dioxide concentration of the atmosphere continuously [1–6]. Measurements indicate that a steady increase in the concentration of CO_2 in the atmosphere has taken place since the industrial revolution [4], possibly as a result of the widespread use of fossil fuels. (The current level is about 340 ppm, compared

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with 260 ppm, the estimated preindustrial level [4].) Carbon dioxide is transparent to incoming solar radiation but nearly opaque to longer wavelength thermal radiation reflected from the earths surface. The trapping of radiation increases the average temperature of the earth. (greenhouse effect) An important theoretical question is whether the observed trend in the increase of CO_2 concentration will continue, oscillate, or reach an equilibrium. To answer this question, it is essential to understand the stability of the biomass–carbon dioxide equilibrium and how it is affected by anthropogenic oxidation of carbonaceous matter and deforestation. In this work we present a simple mathematical model that should help to understand the stability properties of the CO_2–biomass system.

MODEL

The simplest model of the green plant biomass–carbon dioxide system is the following. If C is the carbon dioxide concentration (measured in kg of carbon per m^3) and B is average surface concentration of photosynthetic organisms (measured in kg of carbon per m^2), then biomass increases at a rate proportional to BC. The rate of removal of biomass due to the death of organisms, litter, and respiration is proportional to B. If we assumed that the biomass lost (excluding respiration) is rapidly reoxidized to CO_2 by natural biodegradation, the rate equations describing the process can be written in the form

$$\frac{dB}{dt} = aCB - kB,$$

$$\frac{dc}{dt} = -aCB + kB, \tag{1}$$

where a and k are constants. The equations (1) take into account the conservation of carbon, i.e., $B + C = T$ (a constant). The equilibrium points of (1) are

$$\bar{C} = \frac{k}{a}, \qquad \bar{B} = T - \frac{k}{a}. \tag{2}$$

The stability of the equilibrium (2) can be determined by setting

$$C = \bar{C} + c, \qquad B = \bar{B} + b$$

in (1) and solving it in the linear approximation, which yields

$$b \propto e^{-\lambda t}, \qquad C \propto e^{-\lambda t}, \tag{3}$$

where λ is a positive constant. The exponential decay of the perturbations shows that the equilibrium is stable [8]. The stable solution can be expressed in the form

$$B = \frac{B_0(aT - K)\exp(aT - k)t}{(aT - k) + aB_0[\exp(aT - k)t - 1]},$$
$$C = T - B, \tag{4}$$

where B_0 is a constant. The plot of B versus t is shown in Figure 1. It is seen that the biomass reaches a high equilibrium value from a low initial value, whereas the CO_2 concentration approaches an equilibrium level much lower than the initial concentration. Very crudely the model represents the evolution of the primeval CO_2 rich atmosphere to the CO_2 depleted present day conditions with the proliferation of photosynthetic organisms.

In the model (1) we have assumed that the dead biomass is rapidly recycled into carbon dioxide by oxidation. However, in reality the removal of biomass and oxidation into CO_2 occur at different rates, the former generally being faster because biomass cannot be completely oxidized as soon as it is removed. Again, CO_2 could also be produced independently of the biomass concentration (i.e., to a first approximation the carbon dioxide production rate is linear in B and thus contains a term independent of B). When these factors are included, the rate equations (1) get modified to

$$\frac{dB}{dt} = aCB - kB,$$
$$\frac{dC}{dt} = -aCB + k'B + R, \tag{5}$$

where $k' < k$ and R is a constant. The equilibrium points of (1) are

$$\bar{B} = \frac{R}{k - k'},$$
$$\bar{C} = \frac{k}{a}. \tag{6}$$

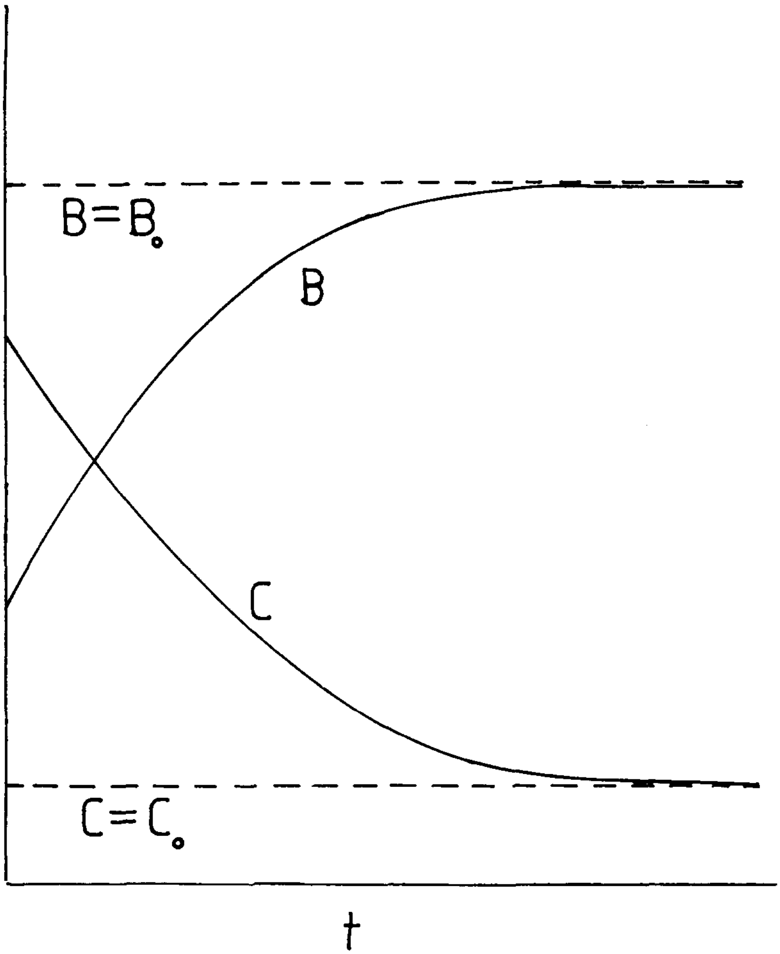

FIG. 1. Biomass (B) and carbon-dioxide (C) generation as predicted by the equations (1).

Linearization of (5) about the point (6) shows that the deviations (b and c) from the equilibrium vary with time according to (3) with

$$\lambda = -\frac{Ra}{k-k'} \pm \frac{Ra}{k-k'}\sqrt{1-\frac{4(k-k')^2}{R^2a^2}} \,. \tag{7}$$

As λ is always negative or has a negative real part, the equilibrium (6) is stable (when λ is negative, the system is asymptotically stable and perturbations from the equilibrium undergo exponential decay, whereas if λ is complex with a negative real part, the system exhibits damped oscillations when disturbed).

The model can be further modified to take into account the biomass removal due to deforestation. If D is the deforestation rate (in units of carbon kg/m^2), we obtain

$$\begin{aligned} \frac{dB}{dt} &= aCB - kB - D, \\ \frac{dC}{dt} &= -aCB + k'B + R, \end{aligned} \tag{8}$$

where the equilibrium points are

$$\begin{aligned} \bar{B} &= \frac{R-D}{k-k'}, \\ \bar{C} &= \frac{Rk - Dk'}{a(R-D)}, \end{aligned} \tag{9}$$

and linearization about the equilibrium point as in (1) or (5) yields

$$\lambda = \left[\frac{D(k-k')}{R-D} - \frac{a(R-D)}{k-k'}\right] \pm \sqrt{\left[\frac{D(k-k')}{R-D} - \frac{a(R-D)}{k-k'}\right]^2 - 4aR}. \tag{10}$$

In contrast with the previous two cases, λ given in (10) could become positive, indicating an instability of the solution.

It is also instructive to consider the limiting case $k = k'$, $R = D$, i.e.,

$$\begin{aligned} \frac{dB}{dt} &= aCB - kB - D, \\ \frac{dC}{dt} &= -aCB + kB + D, \end{aligned} \tag{11}$$

where all carbon in the biomass is rapidly recycled to carbon dioxide. It is easy to show that B continuously decreases with increase of C if

$$D > \frac{(aT - k)^2}{4a}, \tag{12}$$

where $B + C = T$ (constant). When the rate of deforestation is larger than the critical value $D_c = (aT - K^2)/4a$, the live biomass gradually decays, converting almost all carbon into carbon dioxide.

CONCLUSION

This simple model illustrates the conditions under which the biomass–carbon dioxide equilibrium could remain stable. If the removed biomass is rapidly reconverted into carbon dioxide, a stable equilibrium is maintained. Carbon dioxide generation at a constant rate (e.g. from anthropogenic sources) could also lead to a stable situation. However, when the deforestation exceeds a certain limit, the equilibrium becomes unstable with a rapid increase in the carbon dioxide concentration. Thus one could conclude that the control of deforestation is far more important than preventive measures taken to reduce carbon dioxide emissions into the atmosphere.

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