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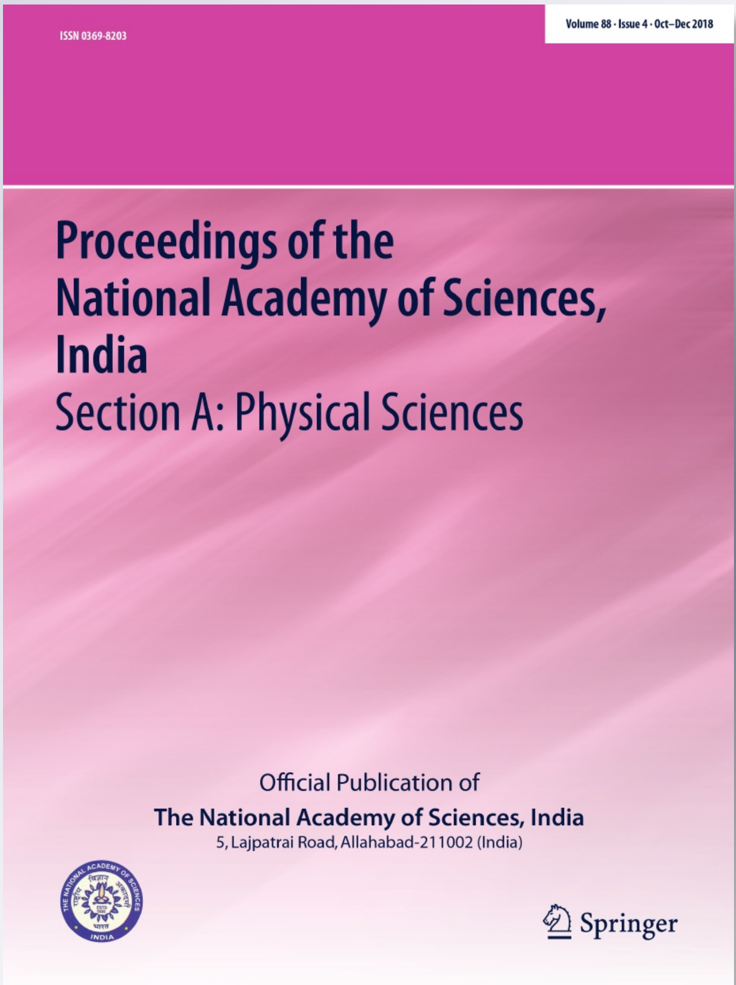

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RESEARCH ARTICLE

SUSY Quantum Mechanics for PT Symmetric Systems

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Abstract A new way of constructing SUSY partner potentials with PT symmetry is proposed. In this construction, PT symmetric superpotentials generate PT symmetric SUSY partners and supercharges that satisfy commutator-anti-commutator relations of close superalgebra SL(1/1). Conversely, every PT symmetric Hamiltonian having zero or non zero ground state energy can be generated using PT symmetric superpotentials. Further, superpotentials having separate P and T symmetries can generate strictly isospectral SUSY partners if the wedge of integration can accommodate two asymptotically opposite quantization contours.

Keywords SUSY · PT-symmetry · Shape invariance potentials · Non-shape invariant potentials · Matrix diagonalization method · Complex space solution

1 Introduction

Lack of evidence for existence of SUSY partners of quarks, leptons and gauge bosons strongly suggest that if SUSY exists, it must have been spontaneously broken in nature. Possible mechanisms of breaking down SUSY are therefore of great interest [1, 2]. In order to study the breaking down of SUSY, conventional supersymmetric quantum mechanics was proposed by Witten [3]. Since conventional quantum mechanics mainly deals with Hermitian Hamiltonians due to reality of their spectra, conventional SUSY quantum mechanics has advanced along the lines of real superpotentials and Hermitian SUSY partners particularly after the work of Gendenshtein [4, 5], who introduced the concept of shape invariance. Later on Oikonomou [6] modified the shape invariance SUSY system using Z_3-graded symmetry using Lie algebra. Further the algebra of Z_3-graded quantum symmetry can lead to some constraints in SUSY QM [6]. Other interesting papers [7–12] on conventional SUSY deals with either reflectionless potential or non-reflectionless potentials, scraf II potential, spectral bifurcation, complex optical potential, square well potential, optical couplers etc.

Bender and Boettcher [13] have introduced PT invariance condition $[H, PT] = 0$ in PT symmetric quantum mechanics, which lifts the requirement of Hermiticity of conventional quantum mechanics and replaces it with unbroken PT symmetry to preserve the reality of quantum spectra. Since its inception, PT symmetric quantum mechanics developed slowly into one of the active areas of research in quantum mechanics. During the last decade there has been an increased interest in non-Hermitian PT-symmetric Hamiltonian systems due to possible applications of non-Hermitian models in particle-physics [14], quantum optics [15], supersymmetric [16], and magneto-hydrodynamics [17] models. PT symmetric Hamiltonians are invariant under space-time reflection: for P, $p \to -p$, $x \to -x$ and for T, $p \to -p$, $x \to x$ and $i \to -i$ and satisfy the commutation relation $[x, p] = i$.

In recent years, several attempts have been made to introduce PT symmetry into the SUSY quantum mechanics

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[18–21] . Andrianov et al. [18] has extended the standard intertwining relations used in SUSY quantum mechanics for non-Hermitian Hamiltonians by introducing complex superpotentials in the place of real superpotentials. In conventional SUSY quantum mechanics, real superpotentials produce Hermitian superpartner Hamiltonians and supercharges which satisfy commutator- anticommutator relations of close superalgebra SL(1/1). However, the extension made by Andrianov et al. [18] does not show such a clear relationship between superpotentials and the corresponding non-Hermitian superpartner Hamiltonians especially in the context of PT symmetry. On the other hand in [19, 20] Supersymmetry has been introduced to non-Hermitian PT symmetric systems by combining conventional SUSY quantum mechanics with the time reversal operator T to create nonstandard creation and annihilation-type operators. These new nonstandard operators preserve the underlying supersymmetric algebra SL(1/1) and extend the concept of the supersymmetric partners to non-Hermitian Hamiltonians. However, in this construction also, relationship between PT symmetric nature of the partner Hamiltonians and the Hermiticity of corresponding superpotential is not clear. In [21], Mostafazadeh has generalized the SUSY quantum mechanics for pseudo-Hermitian Hamiltonians by introducing a z_3-grading operator and an even Hermitian linear automorphism with pseudosuperalgebra. Further recent experimental demonstration of a unidirectional reflectionless parity-time meta material at optical frequencies by Feng et al. [22] and above survey of literature, we feel that something new can be added to this *PT* symmetric systems involving SUSY behavior [1, 2].

2 Generation of PT Symmetric SUSY Partners

First we construct general SUSY theory for PT symmetric Hamiltonians by defining two operators A and B such that

$$A \equiv i\frac{d}{dx} + W \tag{1}$$

and

$$B \equiv i\frac{d}{dx} - W \tag{2}$$

where W is a complex function. Then A does not commute with B ($AB \neq BA$) and unlike in the case of conventional supersymmetric quantum mechanics, A and B are not Hermitian conjugates of each other (*i.e.* $A^\dagger \neq B$ or $B^\dagger \neq A$). Next we define pairs of operators H_+ and H_- as

$$H_+ \equiv AB = -\frac{d^2}{dx^2} - W^2 - i\frac{dW}{dx} \tag{3a}$$

and

$$H_- \equiv BA = -\frac{d^2}{dx^2} - W^2 + i\frac{dW}{dx}. \tag{3b}$$

Obviously H_+ and H_- are not Hermitian when W has non vanishing real and imaginary parts. Further if we impose the condition that W is *PT* symmetric,

$$PTWPT = W. \tag{4}$$

then H_+ and H_- also become PT symmetric (W is *PT* symmetric implies that both W^2 and $i\frac{dW}{dx}$ are also PT symmetric). As in the case of conventional supersymmetric quantum mechanics, we can construct SUSY algebra for PT symmetric systems by defining operators H, Q and $\widetilde{Q}$ as

$$H = \begin{bmatrix} H^- & 0 \\ 0 & H^+ \end{bmatrix}, \tag{5}$$

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \tag{6}$$

and

$$\widetilde{Q} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \tag{7}$$

and they satisfy usual commutation and anticommutation relations of closed superalgebra sl(1/1), $[H, Q] = 0$, $[H, \widetilde{Q}] = 0$, $\left\{Q, \widetilde{Q}\right\} = H$, $\{Q, Q\} = 0$ and $\left\{\widetilde{Q}, \widetilde{Q}\right\} = 0$.

Definitions of A, B, H_- and H_+ show that if the eigenvalue problem can be defined for H_- in some region of the complex plane, then it has a zero energy eigenstate and the function W is not only PT symmetric, but also a superpotential generating H_- and H_+ as PT symmetric SUSY partners. To see this in detail we assume that the eigenvalue problem can be well defined for H_- in some region of the complex plane and define $\phi_0^-(x)$ as

$$\phi_0^-(x) \equiv e^{i\int^x W(t)dt}. \tag{8}$$

Then

$$\frac{d^2\phi_0^-(x)}{dx^2} = \left[-W^2 + i\frac{dW}{dx}\right]\phi_0^-(x) \tag{9}$$

and hence

$$H_-\phi_0^-(x) = 0. \tag{10}$$

In order for $\phi_0^-(x)$ to become a normalizable eigenfunction of H_-, it should satisfy the condition that $\phi_0^-(x) \to 0$ as $|x| \to +\infty$ in some region in the complex plane. First we assume that such region exist and none of the eigenvalues of H_+ is zero. (Later we give the condition for both $H_+$and H_- to have zero energy eigenstates.). By following usual supersymmetric arguments, we can see that eigenvalues and eigenfunctions of H_+ and H_- are related. Let the eigenvalues and eigenfunctions of H_+ and H_- be

$\{E_n^+, \phi_n^+(x)\}$ and $\{E_n^-, \phi_n^-(x)\}$ respectively. Then $E_0^- = 0$ and for $n > 0$,

$$H_+ A\phi_n^-(x) = ABA\phi_n^-(x) = E_n^- A\phi_n^-(x) \quad (11)$$

and

$$H_- B\phi_m^+(x) = BAB\phi_m^+(x) = E_m^+ B\phi_m^+(x) \quad (12)$$

and therefore the eigenvalues and eigenfunctions of the two Hamiltonians H_+ and H_- are related by

$$E_0^- = 0, E_n^+ = E_{n+1}^-, \phi_n^+(x) = A\phi_{n+1}^-(x) \quad (13)$$

and

$$\phi_{n+1}^-(x) = B\phi_n^+(x) \quad (14)$$

for $n = 0, 1, 2, \ldots$ Hence energy levels of H_+ and H_- are supersymmetric and H_+ and H_- are not only *PT* symmetric but also supersymmetric partners. This is the first result of this paper.

3 PT Symmetric Superpotentials and Isospectral Partners

Now we consider a Hamiltonian H which is PT symmetric and has zero energy ground state $\Psi_0(x)$. Then $H\Psi_0 = 0$, and since H is PT symmetric, $PTH\Psi_0 = HPT\Psi_0 = 0$. Therefore $PT\Psi_0$ is also a zero energy eigenstate. If we assume that the ground state is non-degenerate, then $PT\Psi_0 = \lambda\Psi_0$. Since $W(x) = i\frac{d\Psi_0}{dx}/\Psi_0$, $PTW(x) = W(x)$ and hence $W(x)$ is also PT symmetric. Therefore for a given PT symmetric Hamiltonian with zero energy ground state, the corresponding superpotential is PT symmetric and further from (3) and (4) SUSY partner of H is also PT symmetric.

Next we investigate the condition for which H_+ and H_- are strictly isospectral. First we note that when $E_0^- = 0$ and $E_0^+ = 0$, H_+ and H_- are strictly isospectral. If SUSY is not broken, $H_-\phi_0^-(x) = 0$ and $\phi_0^-(x) \to 0$ as $|x| \to +\infty$ in some regions of the complex plane. As in the case of H_-, we express $\phi_0^+(x)$ as

$$\phi_0^+(x) \equiv e^{-i\int^x W(t)dt} \quad (15)$$

and it satisfies the equation

$$H_+\phi_0^+(x) = 0. \quad (16)$$

Therefore, if $\phi_0^+(x) \to 0$ as $|x| \to +\infty$ in the same region in the complex plane where $\phi_0^-(x) \to 0$ as $|x| \to +\infty$ then H_+ and H_- become strictly isospectral. Now suppose the superpotential $W(x)$ of H_+ and H_- is parity invariant $(PW(x) = W(-x) = W(x))$. Let

$$\Omega(x) = i\int^x W(t)dt \quad (17)$$

then

$$\Omega(x) \to -\infty \text{ as } |x| \to +\infty \quad (18)$$

along some directions θ. Since W is PT symmetric and parity invariant,

$$-\Omega(x) = \Omega(-x) + C \quad (19)$$

and

$$\phi_0^+(x) = C' e^{\Omega(e^{i\pi}x)} \quad (20)$$

where $C' = e^C$ is a constant. Therefore

$$\phi_0^+(x) \to 0 \quad \text{as} \quad |x| \to +\infty \quad (21)$$

along contours which are asymptotically opposite to the directions of θ. If these directions are also inside the wedge of integration then H_+ and H_- are strictly isospectral.

Next we give some examples which illustrate results presented in this paper. First system is generated by the complex PT symmetric superpotential

$$W(x) = ix + 1 \quad (22)$$

H_+ and H_- for this system are given by

$$p^2 + (x - i)^2 + 1 \quad (23)$$

and

$$p^2 + (x - i)^2 - 1 \quad (24)$$

respectively. Exact eigenenergies are $E_n^+ = (2n + 2)$ and $E_n^- = 2n$ where $n = 0, 1, 2\ldots.$ The supersymmetric partners H_+ and H_- are PT symmetric and satisfy the conditions $E_n^+ = E_{n+1}^-$ and $E_0^- = 0$. Similarly the complex PT symmetric superpotential

$$W(x) = ix^3 + 1 \quad (25)$$

will produce supersymmetric partner potentials

$$H_+ = p^2 + x^6 - 1 - 2ix^3 + 3x^2 \quad (26)$$

and

$$H_- = p^2 + x^6 - 1 - 2ix^3 - 3x^2 \quad (27)$$

Eigenspectra of these Hamiltonians are obtained using matrix diagonalization method [23] and given in Table 1.

Obviously H_+ and H_- are PT symmetric and It is evident from Table 1 that eigenenergies satisfy the supersymmetric conditions $E_n^+ = E_{n+1}^-$ and $E_0^- = 0$. In the last illustration we consider a PT symmetric superpotential which is invariant under both parity and time reversal symmetry separately. Let

$$W(x) = x^2 \quad (28)$$

then

Table 1 First five eigenenergies of H_- and first four eigenenergies of H_+ calculated using matrix diagonalization method described in [18] with the matrix size 1000 × 1000

Eigenenergies of H_-	Eigenenergies of H_+
0	1.022 551
1.022 551	5.573 211
5.573 211	11.004 746
11.004 746	17.370 778
17.370 778	

Spectra of H_+ and H_- show the SUSY energy conditions

$$PW(x) = W(-x) = W(x), \tag{29}$$

and

$$TW(x) = W(x), \tag{30}$$

$$H_+ = p^2 - x^4 - 2ix \tag{31}$$

and

$$H_- = p^2 - x^4 + 2ix. \tag{32}$$

Let $x = re^{i\theta}$. Eigenspectra of this system is obtained by numerical integration of Schroedinger equation [24] asymptotically along the lines $\theta_1 = -\pi/6$ and $\theta_2 = -5\pi/6$ for H_- and $\theta_3 = \pi/6$ and $\theta_4 = -7\pi/6$ for H_+ . Eigenspectra of both H_+ and H_- are shown in Table 2.

It is important to note that eigenvalue problems of both H_+ and H_- can be defined in either stoke's wedge and $\theta_3 - \theta_2 = \theta_1 - \theta_4 = \pi$. Hence the strict isospectral conditions for partner potentials have been satisfied.

4 Shape Invariance Potential: SUSY EC

For this, we consider two different types of superpotential to generate SUSY EC or Iso-EC. Here we consider a modification of superpotential given in ref. [25, 26] *i.e*

$$W(x) = ix - \frac{i\lambda}{x} \tag{33}$$

which has been used later in a modified form by Bazeia et al. [27]. In this case we notice

$$PTW(ix)PT = W(ix) = ix - \frac{i\lambda}{x} \tag{34}$$

Here the generalized Hamiltonian becomes

$$H_+ = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda+1)}{2x^2} - \lambda + 0.5 \tag{35}$$

and

$$H_- = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda-1)}{2x^2} - \lambda - 0.5 \tag{36}$$

In this case, we get

Table 2 Energy levels of the isospectral Hamiltonians $H_+ = p^2 - x^4 - 2ix$ and $H_- = p^2 - x^4 + 2ix$

Energy level	$E_n^{(+)}$	$E_n^{(-)}$
0	0	0
1	3.398 150	3.398 150
2	8.700 453	8.700 453
3	14.977 808	14.977 808
4	21.999 601	21.999 601

$$E_n^+ = 2n + 1.5 + 0.5\sqrt{1 + 4\lambda(\lambda+1)} - \lambda \tag{37a}$$

and

$$E_n^- = 2n + 0.5 + 0.5\sqrt{1 + 4\lambda(\lambda-1)} - \lambda \tag{37b}$$

For any value of $\lambda \geq 2$, the energy levels satisfy the SUSY energy conditions as

$$E_n^+(\lambda \geq 2) = 2n + 2 \tag{37c}$$

and

$$E_n^-(\lambda \geq 2) = 2n \tag{37d}$$

with

$$E_0^-(\lambda \geq 2) = 0 \tag{37e}$$

Here interesting point is that the eigenvalues are independent of λ.

5 Shape Invariance Potential and Isospectral Condition

Here we consider the same potential but replaced λ by $-\lambda$. The resulting superpotentials become

$$W(ix) = ix + \frac{i\lambda}{x} \tag{38}$$

Here

$$H_+ = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda-1)}{2x^2} + \lambda + 0.5 \tag{39}$$

and

$$H_- = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda+1)}{2x^2} + \lambda - 0.5 \tag{40}$$

In this case, one can see that

$$E_n^+(\lambda \geq 2) = E_n^-(\lambda \geq 2) = 2n + 2\lambda + 1 \tag{41}$$

as

$$E_n^+ = 2n + 0.5 + 0.5\sqrt{1 + 4\lambda(\lambda+1)} + \lambda \tag{42}$$

and

$$E_n^- = 2n + 1.5 + 0.5\sqrt{1 + 4\lambda(\lambda - 1)} + \lambda \tag{43}$$

Here the energy levels are parameter dependent. we show that by simply changing of λ by $-\lambda$ one can achieve SUSY-EC →Iso-EC.

6 Comparison

After completion of this work, we came across the SUSY PT symmetric proposition of Bazeia et al. [27]. We discuss below the basic difference between the two SUSY theories:

(i) Previous operators dependent [27]:

In the work of Bazeia et al. [27], the operators are dependent:

$$A_1 = \frac{1}{\sqrt{2}}(p + W^+(ix)) = \frac{1}{\sqrt{2}}(p + W(-ix)). \tag{44}$$

$$B_1 = \frac{1}{\sqrt{2}}(p - W(ix)). \tag{45}$$

and constitute the SUSY partners as

$$H_+ = A_1^+ B_1 \tag{46}$$

$$H_- = B_1 A_1^+ \tag{47}$$

Here A_1 and B_1 are generated from each other through the parity operator (P) as

$$A_1 = -PB_1P = p + W(-ix) \tag{48}$$

(ii) Present operators independent

Here operators A and B are independent where

$$A = i\frac{d}{dx} + W(ix)$$

and

$$B = i\frac{d}{dx} - W(ix)$$

constitute the SUSY partner as

$$H_+ = AB \quad \text{and} \quad H_- = BA$$

In the present work, no such relation has been made or no such relation can be found out *i.e.* $-PBP \neq A$. In fact our choice of W is PT invariant as stated in Eq. (4).

(iii) PT symmetric Hamiltonian generated through similarity transformation by Bazeia et al. [27]

PT symmetric Hamiltonian is related to a Hermitian Hamiltonian through a similarity transformation as

$$H = shs^{-1} \tag{49a}$$

where

$$h = a^+ a \tag{49b}$$

and

$$a = \frac{(p - iW(x))}{\sqrt{2}} \tag{49c}$$

considering Harmonic oscillator as an example, one can notice that

$$h = \frac{(p + ix)(p - ix)}{2} \tag{50}$$

therefore

$$a = \frac{(p - ix)}{\sqrt{2}} \tag{51a}$$

so also the operators

$$B_1 = sas^{-1} \tag{51b}$$

and

$$A_1 = (s^+)^{-1} as^+ \tag{51c}$$

Further authors have stated that if the similarity transformation s (or s^{-1}) does not take a state out of the Hilbert space then the Hamiltonian for the PT symmetric theory would have the same spectrum with the Hermitian Hamiltonian as well as inherit its various nice features. This implies that if one can not find (or face difficulty) generating similarity transformation then it is obvious that A_1 and B_1 can not be constructed.

On the otherhand, we would like to state here that there is no restriction in the present work for generating SUSY Hamiltonians. Further the Hamiltonian generated in our case, can hardly be generated using similarity transformation. The only condition is that it should be PT symmetric.

(iv). Present work:

Zero ground state energy

Here we consider the form of potential as $W(ix) = f(i^k, x^r, C)$, where $k + r =$ even number and $C =$ arbitrary constant, to generate SUSY Hamiltonians having zero ground state energy and non-zero ground state energy.

For Example 1. $W(ix) = ix + 1$ Here $k = 1$, $r = 1$ and $C = 1$ so that $k + r = 2 =$ even number Example 2. $W(ix) = ix^3 + 1$ Here $k = 1$, $r = 3$ and $C = 1$ so that $k + r = 4 =$ even number Example 3: $W(ix) = x^2$ Here $k = 0$, $r = 2$ so that $k + r = 2 =$ even number. In the work of Bazeia *et al.* [27], no such proposition nor discussion has been used in selection of superpotential W. From the best of our knowledge, no literature on SUSY deals with the generation of superpotentials considering the selection of k,r and C in $W(ix) = f(i^k, x^r, C)$. However in the present work, if one will select two PT functions simultaneously like $W(ix) = y^n \pm y^m$ with $y = f(i^k, x^r, C)$: $n = k + r$, $m = k + r$ with $n > m$ and $|n + m| > 0$. In order to give one example, we propose $W(ix) = ix^3 + x^2$ such that $PTWPT = W$ Here $y^n = ix^3$ and $y^m = x^2$ hence $n = 4$ and

$m = 2$. Similarly, if one consider $W(ix) = ix + i\frac{\lambda}{x} = y^n \pm y^m$ then $n = 2$ and $m = 0$. In this formalism, $n = 0$ and $m = 0$ are not allowed simultaneously.

7 Summary

In this paper we present SUSY quantum mechanics for PT symmetric systems in a consistent manner. We showed that PT symmetric SUSY partner potentials can be generated by PT symmetric superpotentials. Further, it was shown that if a PT symmetric Hamiltonian has zero energy eigenstate, then a PT symmetric superpotential can be constructed and it generates a super partner Hamiltonian which is also PT symmetric. Strict isospectral condition for PT symmetric SUSY partner Hamiltonians has been obtained and shown that the superpotentials having separate P and T symmetries can generate strictly isospectral SUSY partners if the wedge of integration can accommodate two asymptotically opposite quantization contours. Further we feel reflectionless potentials with *PT* symmetric term as discussed in isospectral case can be experimentally demonstrated using suitable metamaterial at optical frequencies. In the context of zero groundstate energy referring to iso-spectral models may find new perspective in Quantum Cosmology [28], where $H_{\pm}|\psi> = 0$ plays an important role in finding out tunneling models(with a well behaved wave function ψ, a system H tunnels from nothing). Lastly, we would like to state that present theory is new to the literature of SUSY quantum mechanics.

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