Periodic Disappearance of Entanglement in Coined Quantum Walks.

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Abstract

We have proposed a new global coin (C_s) which can introduce entanglement to 2-D quantum walks through a conditional based criterion. This coin depicts interesting properties when compared to the Grover and Hadamard coins. Particle-particle entanglement and entanglement between coin states and position states are studied for a quantum walk in two dimensions. The Von Neumann entropy is used to quantify the entanglement. Grover coin increases the degree of entanglement between particles and the Hadamard coin maintains the same value as of the initial state. The entanglement created by these newly developed coins C_s^1 and C_s^2 shows an oscillatory behavior whereas the entanglement gradually converges only for the coin C_s^2. This may be due to the conditional property of the coin. Another interesting property of our coin is that the entanglement disappears for time steps in the multiples of 8 for selected initial states.

Keywords: **Quantum Random Walk, Entanglement, Entropy, Hadamard coin, Grover coin**

I. INTRODUCTION

Quantum entanglement is a physical event which occurs between two or more systems that interact in such a way that the quantum state of any of them cannot be described independently from the others. Quantum entanglement has many applications and is necessary for understanding true nature of quantum mechanics. If the system is a pure state, the Von Neumann entropy (S) of the reduced density matrix of either subsystem can be used for quantifying entanglement $E(|\varphi\rangle)$ and is given by

$$E(|\varphi\rangle) = S(\rho_1) = S(\rho_2) = -Tr(\rho_1 \log_2 \rho_1) \qquad (1)$$

where, ρ_1 and ρ_2 are the two density matrices of the subsystems.

If eigen values of the reduced density matrix ρ_1 are λ_is then

$$S(\rho_1) = -\sum_i \lambda_i \log_2 \lambda_i \qquad (2)$$

The Quantum Random Walk (QRW) is a platform among many that can be used to understand entanglement. The combine system that undergoes QRW is a pure state and therefore Von Neumann entropy can be utilized to calculate the degree of entanglement.

QRW is classified into two categories as Discrete Time Walk (DTW) and Continuous Time Walk and we only focus on Discrete Time Walk in this paper. DTW evolves according to operations that are governed by a quantum coin (Coin operator) and a shift operator. Normally the shift operator shifts the particle to the left or to the right along one dimensional line according to the state of the coin. In this paper, each particle in the two particle random walk is restricted to one dimension.

The first operator acting on the particle wave function is the coin operator C which is assumed to be unitary. There are different types of coin operators. The most general unitary representation of the coin operator is given below.

$$C = \begin{pmatrix} \cos\theta & e^{i\lambda_1}\sin\theta \\ e^{i\lambda_2}\sin\theta & -e^{i(\lambda_1+\lambda_2)}\cos\theta \end{pmatrix} \qquad (3)$$

The mostly used coin for investigating QRW is the Hadamard coin (C_H) [1] which is symmetric. On the other hand, the walk generated by the Grover coin [1] is unique and very useful in investigating algorithms and sets of Grover coins are defined for each dimension. The general representation for "D" dimensional ($d = 2D$) Grover coin is C_G^d [1].

$$C_G^d = \begin{pmatrix} 2/d - 1 & 2/d & \cdots & 2/d & 2/d \\ 2/d & 2/d - 1 & \cdots & 2/d & 2/d \\ \vdots & & \ddots & & \vdots \\ 2/d & 2/d & \cdots & 2/d - 1 & 2/d \\ 2/d & 2/d & \cdots & 2/d & 2/d - 1 \end{pmatrix} \qquad (4)$$

In this paper we mainly concentrate on 1-D and 2-D Grover coins in addition to a new coin which will be introduced later.

$$C_G^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (5)$$

$$C_G^4 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \tag{6}$$

Quantum random walk, unlike the Classical random walk (CRW) shows many interesting properties. Since it is believed that the properties of QRW can be utilized to speed up algorithms which cannot be achieved through CRW, one of the main applications of these quantum walks is the quantum algorithms. Although many algorithms have been developed utilizing QRWs, the Grover coin is mainly employed for improving the Grover's search algorithm. Shenvi et al [2] has shown that random walk search algorithm is approximately equivalent to the Grover's search algorithm. Formulation of QRW based on other coins presents interesting opportunities for development of novel quantum algorithms. However, our work here is to investigate quantum entanglement in 2-DQRW due to two new coins constructed with 1-D Grover and 1-D Hadamard coins.

One of the interesting properties of the 2-D Grover coin is localization. According to Inui et al [3] the reason behind this is the degeneration of eigen values of the time evolution operator. Unlike the 2-D Grover coin, the new coin which is introduced below (C_s) shows localization to multiple positions which oscillates with time.

By introducing the concept of entanglement we have devised a new coin C_s [4]. The basic idea of this coin is the motion along the x-direction depends on the coin "A" while depending on the outcome of "A", the second coin which governs the motion of the y-direction is chosen from two possible coins "B_1"and "B_2". This special coin depicts interesting results when Hadamard and Grover coins are selected as "A", "B_1" and "B_2". First we construct the 2-D Hadamard coin by taking the tensor product of two 1-D Hadamard coins.

$$C_{HH} = C_H \otimes C_H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{7}$$

where,C_{HH} is 2-D Hadamard coin.

Now we construct a new 2-D coin by combining four 1-D coinsA_1, A_2, B_1 and B_2.

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \tag{8}$$

$$A_1 = \begin{pmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{pmatrix} \tag{9}$$

$$A_2 = \begin{pmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{pmatrix} \tag{10}$$

$$B_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{11}$$

$$B_2 = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \tag{12}$$

Note that $A = A_1 + A_2$. The new coin C_s is given by

$$C_s = A_1 \otimes B_1 + A_2 \otimes B_2 \tag{13}$$

A detail discussion on the general form of equation (13) is given in [4].

II. QUANTUM WALK IN 2D LATTICE

The probability distribution of the quantum walk depends on the coin and the initial state of the coin. The initial states used in this investigation are given by the following set of equations.

$$|\varphi_-\rangle = \frac{1}{\sqrt{2}}|u,d\rangle - \frac{1}{\sqrt{2}}|d,u\rangle \tag{14}$$

$$|\varphi_l\rangle = \frac{1}{2}|u,u\rangle - \frac{1}{2}|u,d\rangle - \frac{1}{2}|d,u\rangle + \frac{1}{2}|d,d\rangle \tag{15}$$

Since the distribution patterns for the 2-D Hadamard coin and the Grover coin for the above initial states are discussed by Omar et al [5] and Tregenna et al [6], they are not discussed here.

III. ENTANGLEMENT BETWEEN THE TWO PARTICLES(E_{pp})

A single particle two dimensional system is equivalent to two particle one dimensional systems. Therefore entanglement can be studied as entanglement between lattice points or subsystems which describe each particle.

The calculated entanglement between two particles [7] shows unique features for the newly introduced coin (C_s) as shown in FIG. 1- FIG. 4.

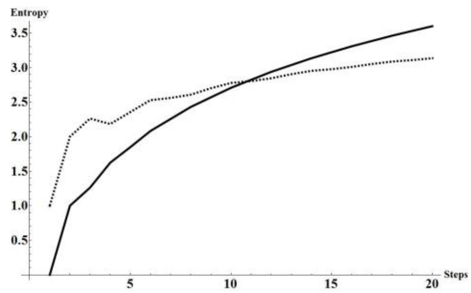

FIG. 1Entropy (E_{pp}) calculated for the Grover coin using initial states $|\varphi_-\rangle$ (Dashed) and $|\varphi_l\rangle$ (Line).

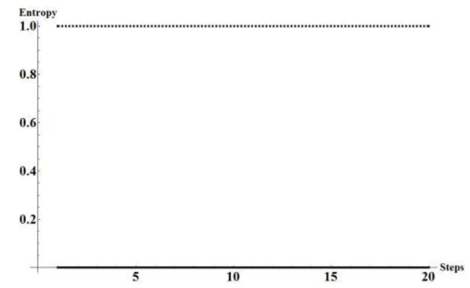

FIG. 2 Entropy (E_{pp}) calculated for the Hadamard coin using initial states $|\varphi_{-}\rangle$ (Dashed) and $|\varphi_{l}\rangle$ (Line).

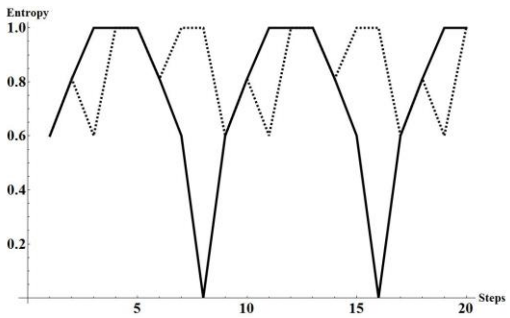

FIG. 3 Entropy (E_{pp}) calculated for the new coin for angles $\alpha = 90^0, \beta = 45^0, \theta = 90^0$in equation (13) using initial states as $|\varphi_{-}\rangle$ (Dashed) and $|\varphi_{l}\rangle$ (Line)

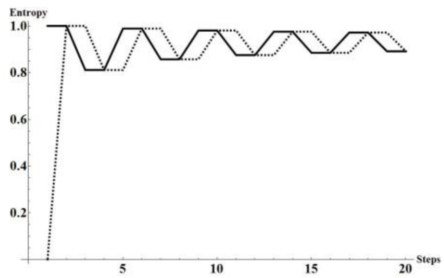

FIG. 4 Entropy (E_{pp}) calculated for the new coin for angles $\alpha = 90^0, \beta = 45^0, \theta = 135^0$ in equation (13) using initial states $|\varphi_{-}\rangle$ (Dashed) and $|\varphi_{l}\rangle$ (Line)

The Grover coin itself increases the entanglement between two particles (FIG. 1) while the Hadamard coin preserves the entanglement (FIG. 2) of the initial coin state. On the other hand the newly introduced coin shows various periodic patterns in the entropy for different choices of initial coin states and various α, β and θ combinations.

The entanglement for the coin of choices $\alpha = 90^0, \beta = 45^0, \theta = 90^0$ (C_s^1) varies among few values periodically as shown in FIG. 3. The entanglement for the initial state $|\varphi_l\rangle$ disappears for time steps which are multiples of 8. However, for the initial state $|\varphi_{-}\rangle$, the degree of entanglement varies among 0.6, 0.8 and 1.

As given in the FIG. 4 the entanglement between the two particles for the coin of choices $\alpha = 90^0, \beta = 45^0, \theta = 135^0$ (C_s^2) shows damping oscillatory motion for $|\varphi_{-}\rangle$ and $|\varphi_l\rangle$ initial states and approaches to a limiting entanglement value between 0.8 and 1.

IV. ENTANGLEMENT BETWEEN COIN AND POSITION. (E_{cp})

Carneiro et al [1] has discussed the entanglement between coin state and position state using the Von Neumann entropy for the Grover and Hadamard coins for various initial states. The entanglement for the Grover and Hadamard coin oscillates about a calculated asymptotic value and converges [1]. The rate of convergence differs from Grover to Hadamard coin.

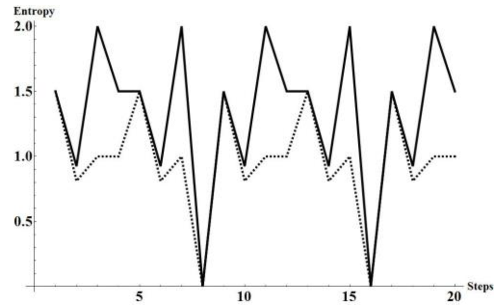

FIG. 5 Entropy (E_{cp}) calculated for the new coin for angles $\alpha = 90^0, \beta = 45^0, \theta = 90^0$in equation (13) using initial states as$|\varphi_{-}\rangle$ (Dashed) and $|\varphi_{l}\rangle$ (Line)

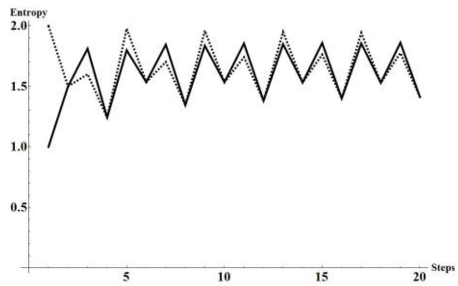

FIG. 6 Entropy (E_{cp}) calculated for the new coin for angles $\alpha = 90^0, \beta = 45^0, \theta = 135^0$ in equation (13) using initial states $|\varphi_{-}\rangle$ (Red) and $|\varphi_{l}\rangle$ (Blue)

The entanglement for the coin C_s^1 oscillates (FIG. 5). As in the case of particle – particle entanglement, the entanglement between position state and coin state for the initial states$|\varphi_l\rangle$ and $|\varphi_{-}\rangle$ disappears for time steps which are multiples of 8. For C_s^2 the entanglement has a damping oscillatory motion for the $|\varphi_l\rangle$ and $|\varphi_{-}\rangle$ initial states (FIG. 6). This shows that the entanglement between the coin state and the particle approaches to a limiting value between 1.4 and 1.8.

V. CONLUSION

The entanglement was quantified using two different criteria, namely entanglement between two particles and entanglement between coin state and position state.

Grover coin increases the degree of entanglement between the particles while the Hadamard coin maintains the same value as of the initial state. The entanglement between coin state and position state behaves differently for the Grover coin and Hadamard coin. It oscillates about a calculated asymptotic value and converges [1].

Newly introduced coins C_s^1 and C_s^2 have special properties. Unlike in the cases of Grover and Hadamard coins, the entanglement created by these two coins demonstrates the oscillatory behavior even between particle and particle. In both entanglement criteria, the degree of entanglement oscillates between values periodically and in the coin C_s^2 it gradually converges. We believe this is mainly due to the conditional nature of our coin which creates a similar behavior in QRW based on the Grover and Hadamard coins by the conditional dependence of shift operator on the outcome of the coin.

REFERENCES

[1] Carneiro, I., Loo, M., Xu, X., Girerd, M., Kendon, V. and Knight, P.L., 2005. Entanglement in coined quantum walks on regular graphs. *New Journal of Physics*, *7*(1), p.156.

[2] Shenvi, N., Kempe, J. and Whaley, K.B., 2003. Quantum random-walk search algorithm. *Physical Review A*, *67*(5), p.052307.

[3] Inui, N., Konishi, Y. and Konno, N., 2004. Localization of two-dimensional quantum walks. *Physical Review A*, *69*(5), p.0523-23.

[4] Nanayakkara, A., Dodangodage, R. and Jayakody, M.N. Entanglement in 2D Quantum Walks.(*Submitted for publication*)

[5] Omar, Y., Paunković, N., Sheridan, L. and Bose, S., 2006. Quantum walk on a line with two entangled particles. *Physical Review A*, *74*(4), p.042304.

[6] Tregenna, B., Flanagan, W., Maile, R. and Kendon, V., 2003. Controlling discrete quantum walks: coins and initial states. *New Journal of Physics*, *5*(1), p.83.

[7] Berry, S.D. and Wang, J.B., 2011. Two-particle quantum walks: Entanglement and graph isomorphism testing. *Physical Review A*, *83*(4), p.042317.